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FORMULATION OF THE EQUATIONS OF MOTION OF A DEFORMABLE AIRCRAFT USING LAGRANGE'S EQUATIONS IN AN ARBITRARY NON-INERTIAL FRAME OF REFERENCE

by

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December 1978

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9 Technical ~~Memorandum~~ Structures 941

Received for printing 5 December 1978

6 # FORMULATION OF THE EQUATIONS OF MOTION OF A DEFORMABLE AIRCRAFT USING LAGRANGE'S EQUATIONS IN AN ARBITRARY NON-INERTIAL FRAME OF REFERENCE

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10 D. L. Woodcock

SUMMARY

The equations of motion of a deformable aircraft are developed in detail from Lagrange's equations for a non-inertial frame. Particular account is taken of the influence of the propulsive and effective forces produced by power units containing rotating parts. The development ventures to a certain extent into the non-linear regime. By an appropriate choice of deformation modes the principal frame of reference can be specified as, for example, mean-body axes or body-fixed axes.

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1 INTRODUCTION

There have been a number of recent papers on the development of equations of motion of deformable aircraft (*eg* Refs 1-5 and 8-10). One may well ask isn't that enough? To help the reader form his opinion, we state here the purpose of the present paper. It is briefly, to

(i) derive in detail the equations of motion of a deformable aircraft using an arbitrary non-inertial frame as the frame of reference for Lagrange's equations*;

(ii) develop these equations up to and including all second order terms in the deformational and body freedom coordinates;

(iii) take account in the equations of motion of the fact that part, or parts of the propulsive unit may be rotating.

In the past when a non-inertial frame of reference has been used, it has almost without exception been either body-fixed axes (origin and orientation fixed in a small portion of the aircraft) or mean-body axes (origin at the centre of gravity and orientated so that the kinetic energy relative to the axes is a minimum). The deformation modes had therefore to satisfy certain particular conditions. In Ref 2, the equations of motion were in effect obtained for an arbitrary non-inertial frame by transformation from those obtained using body-fixed axes but they are slightly different** from those here obtained as a consequence of a different representation of the deformations. Here we assume the deformations relative to the arbitrary non-inertial frame are precisely a linear function of the generalised coordinates while in Ref 2 it was just the deformations relative to the body-fixed axes which had this characteristic. In Ref 2 the forms of perturbation, when using either of the non-inertial frames, arbitrary or body-fixed, were selected so that any perturbation which could be described by the one, could also be precisely described by the other.

Non-linear effects is a convenient phrase when one has discrepancies between predictions and experience, but few have made any attempt to include non-linearities in their mathematical model, and when they have, it has only been in a restricted way - non-linear behaviour of a power control unit, for example. To aid any more general non-linear investigation it did therefore seem

* Alternatively one can think of this as 'using a non-inertial frame ... in conjunction with an arbitrary set of deformation modes.

** The linear approximations to the equations of motion, though not the same, are exactly equivalent since the linear approximations to the perturbations are the same.

worthwhile to carry out the present development of the equations of motion as far as the second order terms in the generalised coordinates. However, before one performs any non-linear dynamical analysis one should be firmly convinced that its results will be meaningful. Solution will not be easy in general; and one should ask oneself "Is my data adequate? Are my assumptions about the form of the local forces sufficiently near the truth?" and so on (*cf* Ref 4). In particular, in connection with this work one may well ask what possibly can be gained by the use of the second order equations. They certainly won't make any change in the stability or otherwise of the datum state. They may, however, predict a boundary to the extent of the stability, whereas with the linear system if we have stability then it is always stability 'in the large' (and vice versa). There is little value in having a stable datum state if the extent of the stability is very small, and conversely an 'unstable' datum state may be of little concern if the extent of the instability is very small.

Years ago someone made the assumption "Oh, we can neglect that" - perhaps because it didn't seem worth the effort to do anything else - and their successors have followed in their footsteps without question, or so it appears. One such assumption was to take no account, in aircraft dynamical studies such as stability and flutter, of the fact that the aircraft propulsive unit contains one or more rotating parts having an appreciable moment of inertia. Consideration is sometimes given to non-linear aerodynamic terms which couple symmetric and anti-symmetric freedoms; but may not the linear gyroscopic coupling between the same freedoms, resulting from engine rotation, be just as important? And there lies the the reason for the third aspect of the present development.

These last two aspects - non-linearities and rotating parts - add considerable complication to the development (*cf eg* the size of section 7). Thus, partly for the benefit of those who are mainly interested in the first purpose of the present paper, and also to show how other developments of the equations of motion can be considered as particularisation of the development using an arbitrary non-inertial frame, a second paper[12] has been written omitting the aspects (ii) and (iii).

2 THE SEMI-RIGID MODEL

We wish to derive equations of motion which, to a good approximation, govern perturbations of the motion of an aircraft, from a datum state in which it is flying with constant linear velocity and zero angular velocity in a uniform atmosphere. As a basic frame of reference we will take a 'constant-velocity' frame which is such that, during the datum motion, the aircraft is stationary

relative to it. In addition a second frame of reference is used which is such that the aircraft may be considered as fixed in this frame when undeformed, even during the perturbed motion. This frame is identified therefore by a set of axes called the 'no-deformation-body-fixed' axes'; and deformations are considered, by definition, to be departures from the positions defined to be the undeformed positions within this reference frame. It will be seen therefore that the no-deformation-body-fixed axes can in effect be arbitrarily chosen by the analyst. In other words in going from the unperturbed to the perturbed position of the aircraft he puts down a stepping stone inbetween just where he likes - she (the aircraft) steps rigidly to the stone and from there she deforms herself to make the second step to the perturbed position. It should be appreciated however that the choice of a set of deformation modes does, in effect, select, in the perturbed state, an 'optimum' position for the no-deformation-body-fixed axes. Thus, for example, these axes become mean-body axes if the modes satisfy certain conditions*, and they become body-fixed axes when the modes satisfy another set of conditions*.

Let the constant-velocity frame of reference be identified by constant-velocity axes $0_c x_c y_c z_c$ which are such that, in the datum motion, they coincide with some axes fixed in the aircraft. The location of the no-deformation-body-fixed axes $(0_n x_n y_n z_n)$ can then be specified by the translations and rotations which, when applied to the constant-velocity axes, would make the two sets of axes coincident. Let these translations and rotations be, in succession:

(i) Translations $x_1^{(c)}$, $y_1^{(c)}$, $z_1^{(c)}$ in the directions of the respective constant-velocity axes.

(ii) Successive rotations ψ, θ, ϕ about the carried axes $0_n z$, $0_n y$, $0_n x$.

These translations and rotations will be taken as the six 'body-freedom' generalised coordinates. They uniquely specify the position of the no-deformation-body-fixed axes.

---

* These two sets of conditions are, in terms of the modal matrix R introduced later (equation (2-1)).

(i) $\sum \delta m R = \sum \delta m A_{x_f} R = 0$ for mean-body axes.

(ii) At $x_f = y_f = z_f = 0$, $R = 0$, $\dfrac{\partial R}{\partial x_f}$ has null second and third rows, $\dfrac{\partial R}{\partial y_f}$ has null first and third rows, $\dfrac{\partial R}{\partial z_f}$ has null first and second rows - for body-fixed axes.

We will assume that the second step, the deformations, can be written as a linear combination of n deformational degrees of freedom; and so the position of a particle relative to the no-deformation-body-fixed axes is given by

$$\begin{bmatrix} x_n^{(n)} \\ y_n^{(n)} \\ z_n^{(n)} \end{bmatrix} = \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + R \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \tag{2-1}$$

where R is a modal matrix whose elements are functions of the unperturbed coordinates (x_f, y_f, z_f) of the particle. Note that, in doing so, we are not making any assumption that the deformations are small. We are however excluding the exact representation, in one degree of freedom, of the rotation of a part of the aircraft as a rigid body, but this should be of minor importance (*cf* the treatment of engine rotation in section 7).

Thus the position of a typical particle relative to the origin of the constant-velocity axes and referred to those axes is (*cf* Appendix A of Ref 1)

$$\begin{bmatrix} x_c^{(c)} \\ y_c^{(c)} \\ z_c^{(c)} \end{bmatrix} = \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} + S^T \begin{bmatrix} x_n^{(n)} \\ y_n^{(n)} \\ z_n^{(n)} \end{bmatrix} \tag{2-2}$$

where the axes transformation matrix[1] S is given by

$$S = R_\phi P_\theta Y_\psi \tag{2-3}$$

$$R_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \tag{2-4}$$

$$P_\theta = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{2-5}$$

$$Y_\psi = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2-6}$$

and is such that

$$S^T = S^{-1} \quad . \tag{2-7}$$

If now we substitute in (2-2) the expansion[2] of S :

$$S = I - A_\phi + B_{\phi\theta} + \ldots \tag{2-8}$$

where

$$A_\phi = \begin{bmatrix} 0 & -\psi & \theta \\ \psi & 0 & -\phi \\ -\theta & \phi & 0 \end{bmatrix} \tag{2-9}$$

$$B_{\phi\theta} = \begin{bmatrix} -\tfrac{1}{2}(\psi^2+\theta^2) & 0 & 0 \\ \phi\theta & -\tfrac{1}{2}(\phi^2+\psi^2) & 0 \\ \phi\psi & \theta\psi & -\tfrac{1}{2}(\phi^2+\theta^2) \end{bmatrix} \tag{2-10}$$

etc*

then (2-2) becomes

$$\begin{bmatrix} x_c^{(c)} \\ y_c^{(c)} \\ z_c^{(c)} \end{bmatrix} = \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + \begin{bmatrix} R & I & -A_{x_f} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} + A_\phi R \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + B^T_{\phi\theta} \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + \ldots \tag{2-11}$$

---

* $B_{\phi\theta} = \frac{1}{2}\left\{A_\phi^2 + A_{\left(J_\phi \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}\right)}\right\}$ where J_ϕ is defined by equation (3-4).

where
$$\begin{bmatrix} q_{n+1} \\ q_{n+2} \\ q_{n+3} \end{bmatrix} = \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} \tag{2-12}$$

and
$$\begin{bmatrix} q_{n+4} \\ q_{n+5} \\ q_{n+6} \end{bmatrix} = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} . \tag{2-13}$$

3 VELOCITIES

Writing the linear velocity components of the constant-velocity axes, referred to those axes, as $\{u_f, v_f, w_f\}$, and remembering that their angular velocity is zero, it is easily seen (*cf* Ref 1) that the linear and angular velocities of the no-deformation-body-fixed axes are

$$\begin{bmatrix} u^{(n)} \\ v^{(n)} \\ w^{(n)} \end{bmatrix} = S\begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} + S\begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} \tag{3-1}$$

$$\begin{bmatrix} p^{(n)} \\ q^{(n)} \\ r^{(n)} \end{bmatrix} = Q_\phi\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{3-2}$$

where
$$Q_\phi = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix}$$

$$= I - J_\phi - \tfrac{1}{2}J_\phi^T J_\phi + \dots \tag{3-3}$$

where
$$J_\phi = \begin{bmatrix} 0 & 0 & \theta \\ 0 & 0 & -\phi \\ 0 & \phi & 0 \end{bmatrix} . \tag{3-4}$$

Consequently, using (2-8), the linear velocity of a particle is given by

$$
\begin{aligned}
\begin{bmatrix} u_m^{(n)} \\ v_m^{(n)} \\ w_m^{(n)} \end{bmatrix} &= S \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} + S \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} + R \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} + A_{p^{(n)}} \left\{ \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + R \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right\} \\
&= \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} + \begin{bmatrix} R & I & -A_{x_f} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_{n+6} \end{bmatrix} + A_{u_f} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \\
&\quad + B_{\phi\theta} \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} + A_{\dot{\phi}} R \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + A_{\dot{x}_1^{(c)}} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + A_{x_f} J_{\phi} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \\
&\quad + \dots \quad . \qquad (3\text{-}5)
\end{aligned}
$$

The above expression assumes that the modal matrix R and the unperturbed coordinates (x_f, y_f, z_f) are constant. However when we subsequently come to consider the effect of a rotating part of an engine (section 7.1) this will not be so. The appropriate expression for the velocity of a particle then is

$$\begin{bmatrix} u_m^{(n)} \\ v_m^{(n)} \\ w_m^{(n)} \end{bmatrix} = S \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} + S \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} + \begin{bmatrix} \dot{x}_n^{(n)} \\ \dot{y}_n^{(n)} \\ \dot{z}_n^{(n)} \end{bmatrix} + A_{p^{(n)}} \begin{bmatrix} x_n^{(n)} \\ y_n^{(n)} \\ z_n^{(n)} \end{bmatrix}$$

$$= \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} + \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} + \begin{bmatrix} \dot{x}_n^{(n)} \\ \dot{y}_n^{(n)} \\ \dot{z}_n^{(n)} \end{bmatrix} - A_{x_n^{(n)}} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} + A_{u_f} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

$$+ B_{\phi\theta} \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} - A_{\phi} \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} + A_{x_n^{(n)}} J_{\phi} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \quad . \qquad (3\text{-}6)$$

From (3-5) we find that, apart from points on such a rotating part,

$$\begin{bmatrix} \dot{u}_m^{(n)} \\ \dot{v}_m^{(n)} \\ \dot{w}_m^{(n)} \end{bmatrix} + A_{p^{(n)}} \begin{bmatrix} u_m^{(n)} \\ v_m^{(n)} \\ w_m^{(n)} \end{bmatrix} = \ddot{S}^T \left(\begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + R \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right) + S \begin{bmatrix} \ddot{x}_1^{(c)} \\ \ddot{y}_1^{(c)} \\ \ddot{z}_1^{(c)} \end{bmatrix} + 2S\dot{S}^T R \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} + R \begin{bmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_n \end{bmatrix}$$

$$= R \begin{bmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_n \end{bmatrix} - A_{x_f} \begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} + \begin{bmatrix} \ddot{x}_1^{(c)} \\ \ddot{y}_1^{(c)} \\ \ddot{z}_1^{(c)} \end{bmatrix} + A_{\ddot{\phi}} R \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} - A_{\phi} \begin{bmatrix} \ddot{x}_1^{(c)} \\ \ddot{y}_1^{(c)} \\ \ddot{z}_1^{(c)} \end{bmatrix}$$

$$+ 2A_{\dot{\phi}} R \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} + A_{x_f} J_{\phi} \begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} + 2B^T_{\dot{\phi}\dot{\theta}} \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix}$$

$$+ \ldots \quad . \qquad (3\text{-}7)$$

This expression will be required subsequently (section 4.2) in determining the effective forces.

4 EQUATIONS OF MOTION

4.1 For the deformational degrees of freedom

The non-inertial form of Lagrange's equation is[1,6]

$$\frac{\partial V_0}{\partial q_i} + J_i + G_i + \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{q}_i}\right) - \frac{\partial W}{\partial q_i} = \bar{Q}_i \tag{4-1}$$

where V_0 is the centrifugal potential function

W is the kinetic energy relative to the frame of reference

G_i is the gyrostatic force

J_i is a rather similar force to G_i but depending on the angular accelerations rather than the angular velocities of the frame of reference

$\bar{Q}_i$ is the generalised force obtained by the method of virtual work - in the assumed infinitesimal virtual displacement the frame of reference is regarded as stationary

q_i is the generalised coordinate of the ith degree of freedom where that freedom is one such that the position and orientation of the frame of reference is independent of it.

Thus, this equation can be used for the degrees of freedom $i = 1, \ldots, n$ when taking the frame of reference to be the no-deformation-body-fixed axes. The various quantities listed above are given by the following expressions, where δm is the mass of a particle of the aircraft, and the summations are for all particles:

$$V_0 = \sum \delta m \left[\tfrac{1}{2}\begin{bmatrix} p^{(n)} & q^{(n)} & r^{(n)} \end{bmatrix} A^2_{x_n^{(n)}} \begin{bmatrix} p^{(n)} \\ q^{(n)} \\ r^{(n)} \end{bmatrix} - \begin{bmatrix} p^{(n)} & q^{(n)} & r^{(n)} \end{bmatrix} A_{x_n^{(n)}} \begin{bmatrix} u^{(n)} \\ v^{(n)} \\ w^{(n)} \end{bmatrix} + \begin{bmatrix} x_n^{(n)} & y_n^{(n)} & z_n^{(n)} \end{bmatrix} \left\{ \dot{u}^{(n)} \;\; \dot{v}^{(n)} \;\; \dot{w}^{(n)} \right\} \right] \quad \ldots\ldots \tag{4-2}$$

$$W = \tfrac{1}{2}\sum \delta m \left(\dot{x}_n^{(n)2} + \dot{y}_n^{(n)2} + \dot{z}_n^{(n)2} \right) \tag{4-3}$$

$$G_i = 2\sum \delta m \begin{bmatrix} \dfrac{\partial x_n^{(n)}}{\partial q_i} & \dfrac{\partial y_n^{(n)}}{\partial q_i} & \dfrac{\partial z_n^{(n)}}{\partial q_i} \end{bmatrix} A_{p}(n) \begin{bmatrix} \dot{x}_n^{(n)} \\ \dot{y}_n^{(n)} \\ \dot{z}_n^{(n)} \end{bmatrix} \qquad (4\text{-}4)$$

$$J_i = \sum \delta m \begin{bmatrix} \dfrac{\partial x_n^{(n)}}{\partial q_i} & \dfrac{\partial y_n^{(n)}}{\partial q_i} & \dfrac{\partial z_n^{(n)}}{\partial q_i} \end{bmatrix} A_{\dot{p}}(n) \begin{bmatrix} x_n^{(n)} \\ y_n^{(n)} \\ z_n^{(n)} \end{bmatrix} . \qquad (4\text{-}5)$$

Using the expressions for the velocities of no-deformation-body-fixed axes (equations (3-1) and (3-2)) and for the location of a particle relative to these axes (equation (2-1)) we find that the centrifugal potential function is given, after some manipulation (*cf* Ref 4, Appendix), by

$$V_0 = -\tfrac{1}{2}[\dot{\phi}\ \dot{\theta}\ \dot{\psi}]\, I_n \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} + [q_1\ \ldots\ q_n]\left(\sum \delta m R^T\right) \begin{bmatrix} \ddot{x}_1^{(c)} \\ \ddot{y}_1^{(c)} \\ \ddot{z}_1^{(c)} \end{bmatrix} + [\dot{\phi}\ \dot{\theta}\ \dot{\psi}]\, J_{\phi} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$+ \left\{ \begin{bmatrix} \ddot{x}_1^{(c)} & \ddot{y}_1^{(c)} & \ddot{z}_1^{(c)} \end{bmatrix} A_{\phi} + 2[\dot{\phi}\ \dot{\theta}\ \dot{\psi}]\, D_{\phi} C_{u_f} \right\} \sum \delta m R \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$+ [\dot{\phi}\ \dot{\theta}\ \dot{\psi}] \left\{ D_{\phi}\left(\sum \delta m C_{x_f} R\right) - J_{\dot{\phi}}^T \left(\sum \delta m \left(J_{x_f} - K_{x_f}\right) R\right) \right\} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$+ \ldots \qquad (4\text{-}6)$$

where $I_n \left(= -\sum \delta m A_{x_f}^2 \right)$ is the matrix of moments and products of inertia of the aircraft (in the datum motion state),

and where

$$D_\phi = \mathrm{diag}\{\phi \;\theta\; \psi\} \tag{4-7}$$

$$K_\phi = \begin{bmatrix} 0 & -\psi & 0 \\ \psi & 0 & 0 \\ -\theta & 0 & 0 \end{bmatrix} = A_\phi - J_\phi \tag{4-8}$$

$$C_{u_f} = \begin{bmatrix} 0 & v_f & w_f \\ u_f & 0 & w_f \\ u_f & v_f & 0 \end{bmatrix} \quad \text{etc} \quad . \tag{4-9}$$

Consequently

$$\begin{aligned}
\begin{bmatrix} \frac{\partial V_0}{\partial q_1} \\ \vdots \\ \frac{\partial V_0}{\partial q_n} \end{bmatrix} = {} & \left(\sum \delta m R^T\right)\begin{bmatrix} \ddot{x}_1^{(c)} \\ \ddot{y}_1^{(c)} \\ \ddot{z}_1^{(c)} \end{bmatrix} + \left(\sum \delta m R^T\right)\left\{2C_{u_f}^T D_\phi \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} - A_\phi \begin{bmatrix} \ddot{x}_1^{(c)} \\ \ddot{y}_1^{(c)} \\ \ddot{z}_1^{(c)} \end{bmatrix}\right\} \\
& + \left(\sum \delta m R^T C_{x_f}^T\right) D_{\dot\phi} \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} - \left(\sum \delta m R^T \left(J_{x_f}^T - K_{x_f}^T\right)\right) J_{\dot\phi} \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} \\
& + \ldots \quad .
\end{aligned} \tag{4-10}$$

From (2-1), (3-2) and (3-3) the gyrostatic forces are easily seen to be

$$\begin{bmatrix} G_1 \\ \vdots \\ G_n \end{bmatrix} = 2\sum \delta m R^T A_{\dot\phi} R \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + \ldots \quad . \tag{4-11}$$

Similarly, the J_i forces (equation (4-5)) are

$$\begin{bmatrix} J_1 \\ \vdots \\ J_n \end{bmatrix} = -\left(\sum \delta m R^T A_{x_f}\right)\begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} + \sum \delta m R^T A_{\ddot{\phi}} R \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$+ \left(\sum \delta m R^T A_{x_f}\right)\left\{ J_{\dot{\phi}} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} + J_{\phi} \begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} \right\} + \ldots \quad . \qquad (4\text{-}12)$$

The kinetic energy relative to the frame of reference (the no-deformation-body-fixed axes) is from (2-1) and (4-3)

$$W = \tfrac{1}{2} \sum \delta m \, [\dot{q}_1 \; \ldots \; \dot{q}_n] \, R^T R \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} \qquad (4\text{-}13)$$

and so,

$$\frac{\partial W}{\partial q_i} = 0 \qquad \text{for all } i \, , \qquad (4\text{-}14)$$

and

$$\frac{d}{dt}\begin{bmatrix} \dfrac{\partial W}{\partial \dot{q}_1} \\ \vdots \\ \dfrac{\partial W}{\partial \dot{q}_n} \end{bmatrix} = \sum \delta m R^T R \begin{bmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_n \end{bmatrix} \quad . \qquad (4\text{-}15)$$

Consequently, collecting together the various contributions, we find that the column vector of the effective forces (the left-hand side of equation (4-1)) for the degrees of freedom $1 \rightarrow n$, is

$$
\begin{aligned}
&\left[\sum \delta m R^T R \quad \sum \delta m R^T \quad -\sum \delta m R^T A_{x_f}\right]\begin{bmatrix}\ddot{q}_1\\ \vdots \\ \ddot{q}_{n+6}\end{bmatrix}\\
&+ \left(\sum \delta m R^T\right)\left\{2C^T_{u_f} D_\phi \begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} - A_\phi \begin{bmatrix}\ddot{x}_1^{(c)}\\ \ddot{y}_1^{(c)}\\ \ddot{z}_1^{(c)}\end{bmatrix}\right\}\\
&+ \left(\sum \delta m R^T A_{x_f}\right)\left\{J_{\dot{\phi}}\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} + J_\phi \begin{bmatrix}\ddot{\phi}\\ \ddot{\theta}\\ \ddot{\psi}\end{bmatrix}\right\}\\
&+ \left(\sum \delta m R^T C^T_{x_f}\right) D_{\dot{\phi}}\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} - \left(\sum \delta m R^T\left(J^T_{x_f} - K^T_{x_f}\right)\right) J_{\dot{\phi}}\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix}\\
&+ \left(\sum \delta m R^T A_{\ddot{\phi}} R\right)\begin{bmatrix}q_1\\ \vdots\\ q_n\end{bmatrix} + 2\left(\sum \delta m R^T A_{\dot{\phi}} R\right)\begin{bmatrix}\dot{q}_1\\ \vdots\\ \dot{q}_n\end{bmatrix} + \ldots \quad . \qquad (4\text{-}16)
\end{aligned}
$$

Expressions for the various contributions to the generalised forces (the right-hand sides of equation (4-1)) are derived below (sections 5.1, 6.1, 7.1 and 8.1).

4.2 For the body-freedoms

To obtain the additional equations required to form a complete set we make use, as in Refs 1 and 2, of the principles of linear and angular momentum applied respectively in the directions of and about* the no-deformation-body-fixed axes. These give** (see equation (3-7))

---

* Strictly speaking about stationary axes which are instantaneously coincident with the no-deformation-body-fixed axes.

** We have used the relationship $2B_{\phi\theta} = A_\phi^2 + A\left(J_\phi \begin{bmatrix}\phi\\ \theta\\ \psi\end{bmatrix}\right)$. (4-17a)

$$\begin{bmatrix} \bar{X}^{(n)} \\ \bar{Y}^{(n)} \\ \bar{Z}^{(n)} \end{bmatrix} = \sum \delta m \left\{ \begin{bmatrix} \dot{u}_m^{(n)} \\ \dot{v}_m^{(n)} \\ \dot{w}_m^{(n)} \end{bmatrix} + A_{p^{(n)}} \begin{bmatrix} u_m^{(n)} \\ v_m^{(n)} \\ w_m^{(n)} \end{bmatrix} \right\}$$

$$= \left[\left(\sum \delta mR \right) \quad m \quad - \sum \delta mA_{x_f} \right] \begin{bmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_{n+6} \end{bmatrix}$$

$$+ \left(\sum \delta mA_{x_f} \right) J_\phi \begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} + mA_{\ddot{x}_1^{(c)}} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + 2B^T_{\dot{\phi}\dot{\theta}} \sum \delta m \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + \left(\sum \delta mA_{\ddot{\phi}} R \right) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$+ 2 \left(\sum \delta mA_{\dot{\phi}} R \right) \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} + \dots \qquad (4\text{-}17)$$

and

$$\begin{bmatrix} \bar{L}_n^{(n)} \\ \bar{M}_n^{(n)} \\ \bar{N}_n^{(n)} \end{bmatrix} = \sum \delta m \left(ID + A_{p^{(n)}} \right) A_{x_n^{(n)}} \begin{bmatrix} u_m^{(n)} \\ v_m^{(n)} \\ w_m^{(n)} \end{bmatrix}$$

$$= \sum \delta m A_{x_n^{(n)}} \left\{ \begin{bmatrix} \dot{u}_m^{(n)} \\ \dot{v}_m^{(n)} \\ \dot{w}_m^{(n)} \end{bmatrix} + A_{p^{(n)}} \begin{bmatrix} u_m^{(n)} \\ v_m^{(n)} \\ w_m^{(n)} \end{bmatrix} \right\}$$

$$= \left[\sum \delta m A_{x_f} R \quad \sum \delta m A_{x_f} \quad I_n \right] \begin{bmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_{n+6} \end{bmatrix}$$

$$- \left(\sum \delta m A_{x_f} \right) A_\phi \begin{bmatrix} \ddot{x}_1^{(c)} \\ \ddot{y}_1^{(c)} \\ \ddot{z}_1^{(c)} \end{bmatrix} + \left(A_{\dot{\phi}} I_n - I_n J_{\dot{\phi}} \right) \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$- I_n J_\phi \begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} - A_{\ddot{x}_1^{(c)}} \left(\sum \delta m R \right) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$+ \left\{ \sum \delta m \left(2 A_{x_f} A_{\ddot{\phi}} - A_{\ddot{\phi}} A_{x_f} \right) R \right\} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$+ 2 \left(\sum \delta m A_{x_f} A_{\dot{\phi}} R \right) \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} + \left\{ \sum \delta m A_{R \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}} R \right\} \begin{bmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_n \end{bmatrix} + \cdots \qquad (4\text{-}18)$$

Expressions for the various contributions to the overall forces and moments are derived below (sections 5.2, 6.2, 7.2 and 8.2).

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5 AERODYNAMIC FORCES

The boundary condition to be satisfied by the air motion at the surface of the body involves the velocity of the surface normal to itself. The velocity vector of a particle is given by equation (3-5), when referred to the no-deformation-body-fixed axes; and the slope vector of the body surface, referred to the same axis, will (*cf* equation (2-1)) be merely a function of the generalised coordinates $q_1 \rightarrow q_n$. The local aerodynamic force vector will therefore be a function of the three quantities

$$S\left\{\begin{bmatrix}u_f\\ v_f\\ w_f\end{bmatrix}+\begin{bmatrix}\dot{x}_1^{(c)}\\ \dot{y}_1^{(c)}\\ \dot{z}_1^{(c)}\end{bmatrix}\right\}, \quad Q_\phi\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix}, \quad \begin{bmatrix}q_1\\ \vdots\\ q_n\end{bmatrix} \tag{5-1}$$

and their derivatives with respect to time. As in previous papers[1-4,11] we will take a model containing no hereditary constituent. Even so, it is difficult to decide what second order approximation one should take for the local aerodynamic force vector. The linear approximation should clearly have the form, bearing in mind (2-8) and (3-3),

$$\begin{bmatrix}e^{(n)}\\ f^{(n)}\\ g^{(n)}\end{bmatrix} \approx \begin{bmatrix}e_f\\ f_f\\ g_f\end{bmatrix} + \sum_{j=1}^{n}\begin{bmatrix}e_j\\ f_j\\ g_j\end{bmatrix}q_j + \begin{bmatrix}e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}}\\ f_{\dot{x}} & \cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots & \end{bmatrix}\left\{\begin{bmatrix}x_1^{(c)}\\ y_1^{(c)}\\ z_1^{(c)}\end{bmatrix} + A_{u_f}\begin{bmatrix}\phi\\ \theta\\ \psi\end{bmatrix}\right\}$$

$$+ \begin{bmatrix}e_{\dot{\phi}} & e_{\dot{\theta}} & e_{\dot{\psi}}\\ f_{\dot{\phi}} & \cdots\cdots & \\ g_{\dot{\phi}} & \cdots\cdots & \end{bmatrix}\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} \tag{5-2}$$

where the coefficients e_j, $e_{\dot{x}}$, $e_{\dot{\phi}}$ etc, may be differential operators. The problem is in the choice of the second order terms. If there were no deformations one could, with a little superfluity, cater fairly adequately for such terms by the expression (*cf* equations (2-9), (3-4), (4-7) and (4-8))

$$\sum_{\mu,\nu=\phi,\dot{x}_1^{(c)},\dot{\phi}} \left\{ \left[E_1^{(\mu\nu)} D_\mu + E_2^{(\mu\nu)} J_\mu + A_\mu E_4^{(\mu\nu)} + D_\mu E_5^{(\mu\nu)} \right\} \begin{bmatrix} \nu \\ \vdots \end{bmatrix} \right\} \tag{5-3}$$

where*

$$E_1^{(\mu\nu)} = E_1^{(\nu\mu)} \tag{5-4}$$

$$E_5^{(\mu\mu)} = 0 \tag{5-5}$$

and the elements of the E matrices may be differential operators. We note that, if P and Q are constant 3 × 3 matrices, then

$$A_\mu P + D_\mu Q = \begin{bmatrix} Q_{11} & P_{32} & -P_{23} \\ -P_{31} & Q_{22} & P_{13} \\ P_{21} & -P_{12} & Q_{33} \end{bmatrix} D_\mu + \begin{bmatrix} P_{33} & -Q_{13} & Q_{12} \\ Q_{23} & P_{33} & -P_{32} \\ -P_{13} & -P_{23} & P_{22} \end{bmatrix} J_\mu + \begin{bmatrix} P_{22} & -P_{21} & -P_{31} \\ -P_{12} & P_{11} & -Q_{21} \\ -Q_{32} & Q_{31} & P_{11} \end{bmatrix} K_\mu \; . \tag{5-6}$$

In addition to (5-3), we can probably cover any second order terms involving products of, on the one hand, a body freedom coordinate, and on the other a deformational coordinate, by an expression of the form

---

* A term $E_3^{(\mu\nu)} K_\mu \begin{bmatrix} \nu \\ \vdots \end{bmatrix}$ has not been included in (5-3) because $K_\mu \begin{bmatrix} \nu \\ \vdots \end{bmatrix} = -J_\nu \begin{bmatrix} \mu \\ \vdots \end{bmatrix}$.

$$\sum_{\mu=\phi,\dot{x}_1^{(c)},\dot{\phi}} \left\{ E_1^{(\mu)} D_\mu K_1^{(\mu)} + E_2^{(\mu)} J_\mu K_2^{(\mu)} + E_3^{(\mu)} K_\mu K_3^{(\mu)} + A_\mu E_4^{(\mu)} K_4^{(\mu)} + D_\mu E_5^{(\mu)} K_5^{(\mu)} \right\} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \qquad (5\text{-}7)$$

where the $E_i^{(\mu)}$ matrices are 3 × 3 matrices whose elements may be differential operators, and the $K_i^{(\mu)}$ matrices are constant, 3 × n, matrices. Further terms on similar lines could be added to cater for any second order terms involving the product of two deformational coordinates. We will certainly not do this, however. To do so would necessitate the inclusion, for consistency, of similar terms in the expression for the local structural force (see section 8); and the two together would require a mass of coefficients whose prediction or measurement seems a faint possibility in the dim and distant future. As it is (5-3) and (5-7) introduce a further (351 + 45n) matrix elements beyond the (21 + 3n) elements of the linear approximation. The contributions from these terms to the overall and generalised forces moreover admit of no simplification after their original specification (*cf* Ref 4, section 4). We will therefore in the two subsequent sections (5.1 and 5.2) derive the forces resulting from the approximation to the local aerodynamic force vector which is linear in the three quantities (5-1), and leave any extension to the reader if he so desires. This approximation is

$$
\begin{aligned}
\begin{bmatrix} e^{(n)} \\ f^{(n)} \\ g^{(n)} \end{bmatrix} &\approx \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} + \sum_{j=1}^{n} \begin{bmatrix} e_j \\ f_j \\ g_j \end{bmatrix} q_j + \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots\cdots & \end{bmatrix} \left\{ S \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} + (S - I) \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} \right\} \\
&\qquad + \begin{bmatrix} e_{\dot{\phi}} & e_{\dot{\theta}} & e_{\dot{\psi}} \\ f_{\dot{\phi}} & \cdots\cdots\cdots & \\ g_{\dot{\phi}} & \cdots\cdots\cdots & \end{bmatrix} Q_{\phi} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \\
&\approx \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} + \sum_{j=1}^{n} \begin{bmatrix} e_j \\ f_j \\ g_j \end{bmatrix} q_j + \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots\cdots & \end{bmatrix} \left\{ \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} + A_{u_f} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \right\} + \begin{bmatrix} e_{\dot{\phi}} & e_{\dot{\theta}} & e_{\dot{\psi}} \\ f_{\dot{\phi}} & \cdots\cdots\cdots & \\ g_{\dot{\phi}} & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \\
&\qquad + \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots\cdots & \end{bmatrix} \left(J_{u_f}^{T} J_{\phi} - \tfrac{1}{2} C_{u_f}^{T} D_{\phi} + A_{\dot{x}_1^{(c)}} \right) \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} - \begin{bmatrix} e_{\dot{\phi}} & e_{\dot{\theta}} & e_{\dot{\psi}} \\ f_{\dot{\phi}} & \cdots\cdots\cdots & \\ g_{\dot{\phi}} & \cdots\cdots\cdots & \end{bmatrix} J_{\phi} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}
\end{aligned}
\qquad \cdots\cdots \; (5\text{-}8)
$$

where we have made use of the relationship

$$
B_{\phi\theta} \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} = \left(J_{u_f}^{T} J_{\phi} - \tfrac{1}{2} C_{u_f}^{T} D_{\phi} \right) \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \tag{5-9}
$$

in which

$$
C_{u_f} = \begin{bmatrix} 0 & v_f & w_f \\ u_f & 0 & w_f \\ u_f & v_f & 0 \end{bmatrix} . \tag{5-10}
$$

5.1 Generalised aerodynamic forces

With the expression (5-8) for the local aerodynamic force vector, the column vector of the generalised aerodynamic forces is

$$\begin{bmatrix} Q_1 \\ \vdots \\ Q_n \end{bmatrix} \approx \sum R^T \begin{bmatrix} e^{(n)} \\ f^{(n)} \\ g^{(n)} \end{bmatrix}$$

$$\approx \sum R^T \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} + \sum R^T \begin{bmatrix} e_1 & \cdots & e_n \\ f_1 & \cdots\cdots & \\ g_1 & \cdots\cdots & \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} +$$

$$+ \left(\sum R^T \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots & \end{bmatrix} \right) \left\{ \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} + A_{u_f} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \right.$$

$$\left. + \left(J_{u_f}^T J_\phi - \tfrac{1}{2} C_{u_f}^T D_\phi + A_{\dot{x}_1^{(c)}} \right) \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \right\}$$

$$+ \left(\sum R^T \begin{bmatrix} e_{\dot{\phi}} & e_{\dot{\theta}} & e_{\dot{\psi}} \\ f_{\dot{\phi}} & \cdots\cdots & \\ g_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} \right) \left\{ \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} - J_\phi \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \right\} . \qquad (5\text{-}11)$$

5.2 Overall aerodynamic forces

The translational force on the aircraft, and the moment about the origin of the no-deformation-body-fixed axes, resulting from the aerodynamic load distribution (5-8), are respectively

$$
\begin{bmatrix} X^{(n)} \\ Y^{(n)} \\ Z^{(n)} \end{bmatrix} = \sum \begin{bmatrix} e^{(n)} \\ f^{(n)} \\ g^{(n)} \end{bmatrix}
$$

$$
= \sum \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} + \sum \begin{bmatrix} e_1 & \cdots & e_n \\ f_1 & \cdots\cdots & \\ g_1 & \cdots\cdots & \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}
$$

$$
+ \left(\sum \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots & \end{bmatrix} \right) \left\{ \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} + A_{u_f} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \right.
$$

$$
\left. + \left(J_{u_f}^T J_\phi - \tfrac{1}{2} C_{u_f}^T D_\phi + A_{\dot{x}_1^{(c)}} \right) \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \right\}
$$

$$
+ \left(\sum \begin{bmatrix} e_{\dot{\phi}} & e_{\dot{\theta}} & e_{\dot{\psi}} \\ f_{\dot{\phi}} & \cdots\cdots & \\ g_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} \right) \left\{ \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} - J_\phi \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \right\} \qquad (5\text{-}12)
$$

and

$$
\begin{aligned}
\begin{bmatrix} L_n^{(n)} \\ M_n^{(n)} \\ N_n^{(n)} \end{bmatrix}
&= \sum A_{x_n^{(n)}} \begin{bmatrix} e^{(n)} \\ f^{(n)} \\ g^{(n)} \end{bmatrix} \\
&= \sum A_{x_f} \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix}
+ \sum \left(A_{x_f} \begin{bmatrix} e_1 & \cdots & e_n \\ f_1 & \cdots\cdots & \\ g_1 & \cdots\cdots & \end{bmatrix} - A_{e_f} R \right) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}
+ \left(\sum A_{x_f} \begin{bmatrix} e_{\dot x} & e_{\dot y} & e_{\dot z} \\ f_{\dot x} & \cdots\cdots & \\ g_{\dot x} & \cdots\cdots & \end{bmatrix} \right) \times \\
&\qquad \times \left\{ \begin{bmatrix} \dot x_1^{(c)} \\ \dot y_1^{(c)} \\ \dot z_1^{(c)} \end{bmatrix} + A_{u_f} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \left(J_{u_f}^T J_\phi - \tfrac{1}{2} C_{u_f}^T D_\phi + A_{\dot x_1^{(c)}} \right) \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \right\} \\
&\quad + \left(\sum A_{x_f} \begin{bmatrix} e_{\dot\phi} & e_{\dot\theta} & e_{\dot\psi} \\ f_{\dot\phi} & \cdots\cdots & \\ g_{\dot\phi} & \cdots\cdots & \end{bmatrix} \right) \left\{ \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} - J_\phi \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} \right\} \\
&\quad + \sum A_{R \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}} \begin{bmatrix} e_1 & \cdots & e_n \\ f_1 & \cdots\cdots & \\ g_1 & \cdots\cdots & \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \\
&\quad + \left(\sum A_{R \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}} \begin{bmatrix} e_{\dot x} & e_{\dot y} & e_{\dot z} \\ f_{\dot x} & \cdots\cdots & \\ g_{\dot x} & \cdots\cdots & \end{bmatrix} \right) \left\{ \begin{bmatrix} \dot x_1^{(c)} \\ \dot y_1^{(c)} \\ \dot z_1^{(c)} \end{bmatrix} + A_{u_f} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \right\} \\
&\quad + \left(\sum A_{R \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}} \begin{bmatrix} e_{\dot\phi} & e_{\dot\theta} & e_{\dot\psi} \\ f_{\dot\phi} & \cdots\cdots & \\ g_{\dot\phi} & \cdots\cdots & \end{bmatrix} \right) \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} .
\end{aligned}
\tag{5-13}
$$

We will write the above expressions as

$$\begin{bmatrix} X^{(n)} \\ Y^{(n)} \\ Z^{(n)} \end{bmatrix} = \begin{bmatrix} X_f \\ Y_f \\ Z_f \end{bmatrix} + \begin{bmatrix} X_1 & \cdots & X_n \\ Y_1 & \cdots\cdots & \\ Z_1 & \cdots\cdots & \end{bmatrix}\begin{bmatrix} q_1 \\ \vdots \\ q_n\end{bmatrix}$$

$$+ \begin{bmatrix} X_{\dot x} & X_{\dot y} & X_{\dot z} \\ Y_{\dot x} & \cdots\cdots & \\ Z_{\dot x} & \cdots\cdots & \end{bmatrix}\left\{\begin{bmatrix}\dot x_1^{(c)}\\ \dot y_1^{(c)} \\ \dot z_1^{(c)}\end{bmatrix} + A_{u_f}\begin{bmatrix}\phi\\ \theta\\ \psi\end{bmatrix} + \left(J^T_{u_f}J_\phi - \tfrac{1}{2} C^T_{u_f}D_\phi + A_{\dot x_1^{(c)}}\right)\begin{bmatrix}\phi\\ \theta\\ \psi\end{bmatrix}\right\}$$

$$+ \begin{bmatrix}X_\phi & X_\theta & X_\psi\\ Y_\phi & \cdots\cdots & \\ Z_\phi & \cdots\cdots & \end{bmatrix}\left\{\begin{bmatrix}\dot\phi\\ \dot\theta\\ \dot\psi\end{bmatrix} - J_\phi \begin{bmatrix}\dot\phi\\ \dot\theta\\ \dot\psi\end{bmatrix}\right\} \qquad (5\text{-}14)$$

and

$$\begin{bmatrix} L_n^{(n)} \\ M_n^{(n)} \\ N_n^{(n)} \end{bmatrix} = \begin{bmatrix} L_f \\ M_f \\ N_f \end{bmatrix} + \begin{bmatrix} L_1 & \cdots & L_n \\ M_1 & \cdots\cdots & \\ N_1 & \cdots\cdots & \end{bmatrix}\begin{bmatrix} q_1 \\ \vdots \\ q_n\end{bmatrix}$$

$$+ \begin{bmatrix} L_{\dot x} & L_{\dot y} & L_{\dot z} \\ M_{\dot x} & \cdots\cdots & \\ N_{\dot x} & \cdots\cdots & \end{bmatrix}\left\{\begin{bmatrix}\dot x_1^{(c)}\\ \dot y_1^{(c)} \\ \dot z_1^{(c)}\end{bmatrix} + A_{u_f}\begin{bmatrix}\phi\\ \theta\\ \psi\end{bmatrix} + \left(J^T_{u_f}J_\phi - \tfrac{1}{2} C^T_{u_f}D_\phi + A_{\dot x_1^{(c)}}\right)\begin{bmatrix}\phi\\ \theta\\ \psi\end{bmatrix}\right\}$$

$$+ \begin{bmatrix} L_{\dot\phi} & L_{\dot\theta} & L_{\dot\psi} \\ M_{\dot\phi} & \cdots\cdots & \\ N_{\dot\phi} & \cdots\cdots & \end{bmatrix}\left\{\begin{bmatrix}\dot\phi\\ \dot\theta\\ \dot\psi\end{bmatrix} - J_\phi \begin{bmatrix}\dot\phi\\ \dot\theta\\ \dot\psi\end{bmatrix}\right\} + \left(\sum A_{R_{\left[\begin{smallmatrix}q_1\\ \vdots\\ q_n\end{smallmatrix}\right]}} \begin{bmatrix} e_1 & \cdots & e_n \\ f_1 & \cdots\cdots & \\ g_1 & \cdots\cdots & \end{bmatrix}\right)\begin{bmatrix} q_1 \\ \vdots \\ q_n\end{bmatrix}$$

$$+ \left(\sum A_{R_{\left[\begin{smallmatrix}q_1\\ \vdots\\ q_n\end{smallmatrix}\right]}} \begin{bmatrix} e_{\dot x} & e_{\dot y} & e_{\dot z} \\ f_{\dot x} & \cdots\cdots & \\ g_{\dot x} & \cdots\cdots & \end{bmatrix}\right)\left\{\begin{bmatrix}\dot x_1^{(c)}\\ \dot y_1^{(c)} \\ \dot z_1^{(c)}\end{bmatrix} + A_{u_f}\begin{bmatrix}\phi\\ \theta\\ \psi\end{bmatrix}\right\} + \left(\sum A_{R_{\left[\begin{smallmatrix}q_1\\ \vdots\\ q_n\end{smallmatrix}\right]}} \begin{bmatrix} e_{\dot\phi} & e_{\dot\theta} & e_{\dot\psi} \\ f_{\dot\phi} & \cdots\cdots & \\ g_{\dot\phi} & \cdots\cdots & \end{bmatrix}\right)\begin{bmatrix}\dot\phi\\ \dot\theta\\ \dot\psi\end{bmatrix}$$

$$\cdots\cdots \qquad (5\text{-}15)$$

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where

$$\begin{bmatrix} X_f \\ Y_f \\ Z_f \end{bmatrix} = \sum \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} \tag{5-16}$$

$$\begin{bmatrix} X_1 & \cdots & X_n \\ Y_1 & \cdots\cdots & \\ Z_1 & \cdots\cdots & \end{bmatrix} = \sum \begin{bmatrix} e_1 & \cdots & e_n \\ f_1 & \cdots\cdots & \\ g_1 & \cdots\cdots & \end{bmatrix} \tag{5-17}$$

$$\begin{bmatrix} X_{\dot{x}} & X_{\dot{y}} & X_{\dot{z}} \\ Y_{\dot{x}} & \cdots\cdots\cdots & \\ Z_{\dot{x}} & \cdots\cdots\cdots & \end{bmatrix} = \sum \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots\cdots & \end{bmatrix} \tag{5-18}$$

$$\begin{bmatrix} X_{\dot{\phi}} & X_{\dot{\theta}} & X_{\dot{\psi}} \\ Y_{\dot{\phi}} & \cdots\cdots\cdots & \\ Z_{\dot{\phi}} & \cdots\cdots\cdots & \end{bmatrix} = \sum \begin{bmatrix} e_{\dot{\phi}} & e_{\dot{\theta}} & e_{\dot{\psi}} \\ f_{\dot{\phi}} & \cdots\cdots\cdots & \\ g_{\dot{\phi}} & \cdots\cdots\cdots & \end{bmatrix} \tag{5-19}$$

$$\begin{bmatrix} L_f \\ M_f \\ N_f \end{bmatrix} = \sum A_{x_f} \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} \tag{5-20}$$

$$\begin{bmatrix} L_1 & \cdots & L_n \\ M_1 & \cdots\cdots & \\ N_1 & \cdots\cdots & \end{bmatrix} = \sum \left(A_{x_f} \begin{bmatrix} e_1 & \cdots & e_n \\ f_1 & \cdots\cdots & \\ g_1 & \cdots\cdots & \end{bmatrix} - A_{e_f} R \right) \tag{5-21}$$

$$\begin{bmatrix} L_{\dot{x}} & L_{\dot{y}} & L_{\dot{z}} \\ M_{\dot{x}} & \cdots\cdots\cdots & \\ N_{\dot{x}} & \cdots\cdots\cdots & \end{bmatrix} = \sum A_{x_f} \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots\cdots & \end{bmatrix} \tag{5-22}$$

and

$$\begin{bmatrix} L_{\dot\phi} & L_{\dot\theta} & L_{\dot\psi} \\ M_{\dot\phi} & \cdots\cdots & \\ N_{\dot\phi} & \cdots\cdots & \end{bmatrix} = \sum A_{x_f} \begin{bmatrix} e_{\dot\phi} & e_{\dot\theta} & e_{\dot\psi} \\ f_{\dot\phi} & \cdots\cdots & \\ g_{\dot\phi} & \cdots\cdots & \end{bmatrix} \quad . \tag{5-23}$$

6 GRAVITATIONAL FORCES

On each particle of the aircraft there will be, according to the standard assumption, a gravitational force δmg acting in the z_0 direction (vertically downwards) of the normal earth-fixed axes[7]. If the orientation transformation matrix which takes us from these axes to the constant-velocity axes is S_{Φ_f}, where

$$S_{\Phi_f} = R_{\Phi_f} P_{\Theta_f} Y_{\Psi_f} \tag{6-1}$$

(*cf* equations (2-4), (2-5) and (2-6)); then the matrix which performs the orientation transformation from the normal earth-fixed axes to the no-deformation-body-fixed axes is SS_{Φ_f}. Thus the local gravitational force vector referred to the no-deformation-body-fixed axes is

$$\begin{bmatrix} e_g^{(n)} \\ f_g^{(n)} \\ g_g^{(n)} \end{bmatrix} = SS_{\Phi_f} \begin{bmatrix} 0 \\ 0 \\ \delta mg \end{bmatrix} = \delta mg \left\{ \ell_{\Phi_f} + A_{\ell_{\Phi_f}} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} - \left(\tfrac{1}{2} C^T_{\ell_{\Phi_f}} D_\phi - J^T_{\ell_{\Phi_f}} J_\phi \right) \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \ldots \right\} \tag{6-2}$$

where

$$\ell_{\Phi_f} = \begin{bmatrix} -\sin\Theta_f \\ \sin\Phi_f \cos\Theta_f \\ \cos\Phi_f \cos\Theta_f \end{bmatrix} \tag{6-3}$$

is the last column of S_{Φ_f}. The angles Φ_f, Θ_f, Ψ_f that we have introduced in this section are respectively, the angle of bank, the angle of inclination, and the nose-azimuth angle, in the datum motion. Since our datum motion is flight in a straight line, there is no loss in generality in taking Ψ_f to be zero;

and also, for conventional aircraft under normal flying conditions, the angle of bank Φ_f will also be zero. If the velocity of the aircraft referred to normal earth-fixed axes is $\{u_{f0}, v_{f0}, w_{f0}\}$, during the datum motion, then its velocity referred to the constant-velocity axes is

$$\begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} = u_{f0} j_{\Phi_f} + v_{f0} k_{\Phi_f} + w_{f0} \ell_{\Phi_f} \tag{6-4}$$

where, for $\Psi_f = 0$,

$$j_{\Phi_f} = \begin{bmatrix} \cos\Theta_f \\ \sin\Phi_f \sin\Theta_f \\ \cos\Phi_f \cos\Theta_f \end{bmatrix} \tag{6-5}$$

$$k_{\Phi_f} = \begin{bmatrix} 0 \\ \cos\Phi_f \\ -\sin\Phi_f \end{bmatrix} \tag{6-6}$$

and

$$S_{\Phi_f} = \begin{bmatrix} j_{\Phi_f} & k_{\Phi_f} & \ell_{\Phi_f} \end{bmatrix} . \tag{6-7}$$

6.1 Generalised gravitational forces

The column vector of the generalised gravitational forces is

$$-\begin{bmatrix} G_1 \\ \vdots \\ G_n \end{bmatrix} = \sum R^T \begin{bmatrix} e_g^{(n)} \\ f_g^{(n)} \\ g_g^{(n)} \end{bmatrix}$$

$$= g\left(\sum \delta m R^T\right)\left\{\ell_{\Phi_f} + A_{\ell_{\Phi_f}} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} - \left(\tfrac{1}{2} C^T_{\ell_{\Phi_f}} D_\phi - J^T_{\ell_{\Phi_f}} J_\phi\right) \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \ldots \right\} . \tag{6-8}$$

6.2 Overall gravitational forces

Using the expression (6-2) for the local gravitational force vector, it is easily seen that the overall translational forces, and moments about the origin of the no-deformation-body-fixed axes, are, referred to those axes, given by

$$\begin{bmatrix} X_g^{(n)} \\ Y_g^{(n)} \\ Z_g^{(n)} \end{bmatrix} = \sum \begin{bmatrix} e_g^{(n)} \\ f_g^{(n)} \\ g_g^{(n)} \end{bmatrix} = mg \left\{ \ell_{\Phi_f} + A_{\ell_{\Phi_f}} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} - \left(\tfrac{1}{2} C^T_{\ell_{\Phi_f}} D_\phi - J^T_{\ell_{\Phi_f}} J_\phi \right) \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \ldots \right\} \qquad (6\text{-}9)$$

$$\begin{bmatrix} L_{gn}^{(n)} \\ M_{gn}^{(n)} \\ N_{gn}^{(n)} \end{bmatrix} = \sum A_{x_n^{(n)}} \begin{bmatrix} e_g^{(n)} \\ f_g^{(n)} \\ g_g^{(n)} \end{bmatrix} = g \left\{ \left(\sum \delta m A_{x_f} \right) \ell_{\Phi_f} - A_{\ell_{\Phi_f}} \left(\sum \delta m R \right) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + \left(\sum \delta m A_{x_f} \right) A_{\ell_{\Phi_f}} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \left(A_\phi A_{\ell_{\Phi_f}} - A_{\ell_{\Phi_f}} A_\phi \right) \left(\sum \delta m R \right) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} - \left(\sum \delta m A_{x_f} \right) \left(\tfrac{1}{2} C^T_{\ell_{\Phi_f}} D_\phi - J^T_{\ell_{\Phi_f}} J_\phi \right) \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \ldots \right\} . \qquad \ldots\ldots (6\text{-}10)$$

7 PROPULSIVE FORCES

In Refs 2, 3 and 4, a very simple model of the propulsive forces was used; and, in particular, no account was taken of the fact that most aircraft propulsive units contain one or more rotating parts which, in effect, exert significant 'forces' on the system. The opportunity is therefore being taken in the present development to include a more general, more sophisticated, and, it is hoped, potentially more accurate model. The development will be written assuming the engine consists of just one rotating part but the extension to one with a number

of rotating parts, or, indeed, to a number of engines, is obvious and merely laborious.

7.1 The model

We will assume that the rotating part is axisymmetric and almost rigid, and that in the unperturbed state it is rotating with angular velocity p_p about its axis of symmetry.

Let two points on the axis of symmetry be

$$\begin{bmatrix} x_n^{(n)} \\ y_n^{(n)} \\ z_n^{(n)} \end{bmatrix} = \begin{bmatrix} x_{fa} \\ y_{fa} \\ z_{fa} \end{bmatrix} + R_a \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \qquad (a = 1,2) \qquad (7\text{-}1)$$

such that the direction of rotation is right-handed, about the line from point 1 to point 2, if p_p is positive; and, with $x_{pf} = x_{f2} - x_{f1}$ etc, let us write

$$\begin{bmatrix} x_p \\ y_p \\ z_f \end{bmatrix} = \begin{bmatrix} x_{pf} \\ y_{pf} \\ z_{pf} \end{bmatrix} + (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \quad . \qquad (7\text{-}2)$$

Thus (x_p, y_p, z_p) specify the orientation of the axis of the rotating part. We will assume that

$$x_{pf}^2 + y_{pf}^2 \neq 0 \quad . \qquad (7\text{-}3)$$

Since we are assuming the rotating part is almost rigid - to make it completely rigid would, amongst other things, require $R_2 = R_1$ which is an unreasonable limitation* - we make

$$(R_2 - R_1)^T \begin{bmatrix} x_{pf} \\ y_{pf} \\ z_{pf} \end{bmatrix} = 0 \quad . \qquad (7\text{-}4)$$

---

* This is a consequence of our use of a strictly linear expression for the deformations (equation (7-1) above for example) which, while giving appreciable simplification does exclude the exact representation of a rotation of a part of the aircraft as a rigid body.

Then

$$\left(x_p^2 + y_p^2 + z_p^2\right) = \left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right) + [q_1 \ \dots \ q_n]\,(R_2 - R_1)^T(R_2 - R_1)\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \quad \dots\dots \ (7\text{-}5)$$

and so the perturbation in the length of the axis of the rotating part between our two reference points is of second order in the generalised coordinates.

The axes transformation matrix (*cf* equations (2-3) to (2-6)) which achieves the orientation transformation from the no-deformation-body-fixed axes to axes fixed (almost) in the rotating part with the x-axis parallel to the axis of symmetry (axis of rotation) will then be $R_{p_{pt}}\Pi$ where from the form of $R_{p_{pt}}$ (equation (2-4)), Π must be such that

$$\Pi\begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} = \begin{bmatrix} \sqrt{x_p^2 + y_p^2 + z_p^2} \\ 0 \\ 0 \end{bmatrix} \qquad (7\text{-}6)$$

where, here and subsequently, we take $\sqrt{}$ to mean the positive value of the square root. Consequently, since we also require $\Pi^{-1} = \Pi^T$ (*cf* equation (2-7)), we find that*, for $x_p^2 + y_p^2 \neq 0$

$$\Pi = \mathrm{diag}\left\{\frac{1}{\sqrt{x_p^2 + y_p^2 + z_p^2}} \quad \frac{1}{\sqrt{x_p^2 + y_p^2}} \quad \frac{1}{\sqrt{x_p^2 + y_p^2}\,\sqrt{x_p^2 + y_p^2 + z_p^2}}\right\} \times$$

$$\times \begin{bmatrix} x_p & y_p & z_p \\ -y_p & x_p & 0 \\ -x_p z_p & -y_p z_p & x_p^2 + y_p^2 \end{bmatrix} . \qquad (7\text{-}7)$$

---

* There is an arbitrariness about the signs of the last two terms in the diagonal matrix. The one chosen ensures that, when $z_p = 0$, the Π transformation represents just one rotation. When $y_p = 0$ it can also represent just one rotation if x_p is positive. Otherwise it represents two rotations and no more. One could use (7-7) when $x_p^2 + y_p^2 = 0$ but one then has arbitrarily to assign a value to $x_p\Big/\sqrt{x_p^2 + y_p^2}$ $\left(\text{or } y_p\Big/\sqrt{x_p^2 + y_p^2}\right)$.

Expanding Π as a Taylor series about its unperturbed value Π_f we find that, for any vector $\{\xi \ \eta \ \zeta\}$

$$\Pi\begin{bmatrix}\xi\\ \eta\\ \zeta\end{bmatrix} = \Pi_f\begin{bmatrix}\xi\\ \eta\\ \zeta\end{bmatrix} + \begin{bmatrix}\dfrac{\partial\Pi_f}{\partial x_{pf}}\begin{bmatrix}\xi\\ \eta\\ \zeta\end{bmatrix} & \dfrac{\partial\Pi_f}{\partial y_{pf}}\begin{bmatrix}\xi\\ \eta\\ \zeta\end{bmatrix} & \dfrac{\partial\Pi_f}{\partial z_{pf}}\begin{bmatrix}\xi\\ \eta\\ \zeta\end{bmatrix}\end{bmatrix}\begin{bmatrix}x_p - x_{pf}\\ y_p - y_{pf}\\ z_p - z_{pf}\end{bmatrix}$$

$$+ \frac{1}{2}\begin{bmatrix}[x_p - x_{pf} & y_p - y_{pf} & z_p - z_{pf}]B_{\xi 1}\\ [x_p - x_{pf} & y_p - y_{pf} & z_p - z_{pf}]B_{\xi 2}\\ [x_p - x_{pf} & y_p - y_{pf} & z_p - z_{pf}]B_{\xi 3}\end{bmatrix}\begin{bmatrix}x_p - x_{pf}\\ y_p - y_{pf}\\ z_p - z_{pf}\end{bmatrix} + \ldots$$

...... (7-8)

where the $B_{\xi i}$ are symmetric matrices such that the

(11) element of $B_{\xi i}$ = ith element of $\dfrac{\partial^2\Pi_f}{\partial x_{pf}^2}\{\xi \ \eta \ \zeta\}$

(12) element of $B_{\xi i}$ = ith element of $\dfrac{\partial^2\Pi_f}{\partial x_{pf}\partial y_{pf}}\{\xi \ \eta \ \zeta\}$

(13) element of $B_{\xi i}$ = ith element of $\dfrac{\partial^2\Pi_f}{\partial x_{pf}\partial z_{pf}}\{\xi \ \eta \ \zeta\}$

(22) element of $B_{\xi i}$ = ith element of $\dfrac{\partial^2\Pi_f}{\partial y_{pf}^2}\{\xi \ \eta \ \zeta\}$ etc.

Expressions for the various derivatives of Π_f are obtained in the Appendix. Equation (7-8) can be written in terms of the generalised coordinates by the use of (7-2).

The position of a general point on the rotating part, relative to the point $a = 1$ of (7-1), and referred to the axes 'fixed' in the rotating part, will therefore have the form

$$\begin{bmatrix} x_p^{(p)} \\ y_p^{(p)} \\ z_p^{(p)} \end{bmatrix} = R_{P_{pt}} \Pi \left\{ \begin{bmatrix} x_f - x_{f1} \\ y_f - y_{f1} \\ z_f - z_{f1} \end{bmatrix} + (R - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right\} . \tag{7-9}$$

Our assumption of almost rigidity requires that this expression contains no first order terms in the generalised coordinates. Let us assume that there is a matrix B relating the modal function for a general point on the rotating part with that for a point on its axis of rotation by the equation

$$R - R_1 = B^T (R_2 - R_1) \tag{7-10}$$

which will achieve this. Substituting (7-8) and (7-10) in (7-9), making use of (7-2), and equating the first order terms to zero, we obtain

$$R_{P_{pt}} \left\{ \left[\frac{\partial \Pi_f}{\partial x_{pf}} \begin{bmatrix} x_f - x_{f1} \\ y_f - y_{f1} \\ z_f - z_{f1} \end{bmatrix} \quad \frac{\partial \Pi_f}{\partial y_{pf}} \begin{bmatrix} x_f - x_{f1} \\ y_f - y_{f1} \\ z_f - z_{f1} \end{bmatrix} \quad \frac{\partial \Pi_f}{\partial z_{pf}} \begin{bmatrix} x_f - x_{f1} \\ y_f - y_{f1} \\ z_f - z_{f1} \end{bmatrix} \right] + \Pi_f B^T \right\} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = 0 . \tag{7-11}$$

Thus this will be satisfied for any t and any values of the generalised coordinates, if

$$B = - \begin{bmatrix} \begin{bmatrix} x_{pf}^{(n)} & y_{pf}^{(n)} & z_{pf}^{(n)} \end{bmatrix} \dfrac{\partial \Pi_f^T}{\partial x_{pf}} \Pi_f \\ \begin{bmatrix} x_{pf}^{(n)} & y_{pf}^{(n)} & z_{pf}^{(n)} \end{bmatrix} \dfrac{\partial \Pi_f^T}{\partial y_{pf}} \Pi_f \\ \begin{bmatrix} x_{pf}^{(n)} & y_{pf}^{(n)} & z_{pf}^{(n)} \end{bmatrix} \dfrac{\partial \Pi_f^T}{\partial z_{pf}} \Pi_f \end{bmatrix} \tag{7-12}$$

where
$$\begin{bmatrix} x_{pf}^{(n)} \\ y_{pf}^{(n)} \\ z_{pf}^{(n)} \end{bmatrix} = \begin{bmatrix} x_f - x_{f1} \\ y_f - y_{f1} \\ z_f - z_{f1} \end{bmatrix} \tag{7-13}$$

are the unperturbed coordinates of a particle on the rotating part, relative to the point $a = 1$ of (7-1), and referred to the no-deformation-body-fixed axes. We also of course require B to be such that, at the point $a = 2$ of (7-1), equation (7-10) gives $R = R_2$. Now from (7-12) and the results obtained in the Appendix we find that B at this point is not the unit matrix but (*cf* also equation (7-16), (7-50), (7-50a) and (7-50b))

$$I - \frac{1}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)} \begin{bmatrix} x_{pf} \\ y_{pf} \\ z_{pf} \end{bmatrix} \begin{bmatrix} x_{pf} & y_{pf} & z_{pf} \end{bmatrix} .$$

Thus, the condition (7-4), already stated, is also necessary; and so, if this is satisfied, the desired almost rigidity is achieved by requiring that the modal matrix on the rotating part satisfies (7-10) where B is given by (7-12). It is shown in the Appendix that each of the products $\frac{\partial \Pi_f}{\partial x_{pf}} \Pi_f^T$, $\frac{\partial \Pi_f}{\partial y_{pf}} \Pi_f^T$, $\frac{\partial \Pi_f}{\partial z_{pf}} \Pi_f^T$ is a skew symmetric (A type) matrix. The same is therefore true of the products $\frac{\partial \Pi_f^T}{\partial x_{pf}} \Pi_f$ etc, and so (7-12) can be rewritten as

$$B = \begin{bmatrix} \begin{bmatrix} x_{pf}^{(n)} & y_{pf}^{(n)} & z_{pf}^{(n)} \end{bmatrix} \Pi_f^T \dfrac{\partial \Pi_f}{\partial x_{pf}} \\ \begin{bmatrix} x_{pf}^{(n)} & y_{pf}^{(n)} & z_{pf}^{(n)} \end{bmatrix} \Pi_f^T \dfrac{\partial \Pi_f}{\partial y_{pf}} \\ \begin{bmatrix} x_{pf}^{(n)} & y_{pf}^{(n)} & z_{pf}^{(n)} \end{bmatrix} \Pi_f^T \dfrac{\partial \Pi_f}{\partial z_{pf}} \end{bmatrix} . \tag{7-14}$$

Furthermore, making use of the properties of A type matrices and writing (*cf* the Appendix)

$$C = \begin{bmatrix} y_{pf}z_{pf} & -x_{pf}z_{pf} & y_{pf} \\ -z_{pf}x_{pf} & -y_{pf}z_{pf} & -x_{pf} \\ 0 & x_{pf}^2 + y_{pf}^2 & 0 \end{bmatrix} \times$$

$$\times \operatorname{diag}\left\{ \frac{1}{\left(x_{pf}^2 + y_{pf}^2\right)\sqrt{x_{pf}^2 + y_{pf}^2 + z_{pf}^2}} \quad \frac{1}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)\sqrt{x_{pf}^2 + y_{pf}^2}} \quad \frac{1}{\sqrt{x_{pf}^2 + y_{pf}^2}\sqrt{x_{pf}^2 + y_{pf}^2 + z_{pf}^2}} \right\}$$

$$\ldots\ldots \quad (7\text{-}15)$$

we have

$$B = -C\Pi_f A_{x_{pf}^{(n)}} = -C R_{p_p t}^T A_{x_{pf}^{(p)}} R_{p_p t} \Pi_f \quad . \qquad (7\text{-}16)$$

The position of a general point on the rotating part, relative to the origin of the no-deformation-body-fixed axes and referred to those axes thus, from (7-10) and (7-13), has the form

$$\begin{bmatrix} x_n^{(n)} \\ y_n^{(n)} \\ z_n^{(n)} \end{bmatrix} = \begin{bmatrix} x_{f1} \\ y_{f1} \\ z_{f1} \end{bmatrix} + \begin{bmatrix} x_{pf}^{(n)} \\ y_{pf}^{(n)} \\ z_{pf}^{(n)} \end{bmatrix} + \left\{R_1 + B^T(R_2 - R_1)\right\} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \quad . \qquad (7\text{-}17)$$

Note that the $x_{pf}^{(n)}$ etc, and hence B, are not constant with respect to time for (*cf* (7-9))

$$\begin{bmatrix} x_{pf}^{(n)} \\ y_{pf}^{(n)} \\ z_{pf}^{(n)} \end{bmatrix} = \Pi_f^T R_{p_p t}^T \begin{bmatrix} x_{pf}^{(p)} \\ y_{pf}^{(p)} \\ z_{pf}^{(p)} \end{bmatrix} \qquad (7\text{-}18)$$

where the $x_{pf}^{(p)}$ etc are constant. Thus

$$\begin{bmatrix} \dot{x}_n^{(n)} \\ \dot{y}_n^{(n)} \\ \dot{z}_n^{(n)} \end{bmatrix} = \begin{bmatrix} \dot{x}_{pf}^{(n)} \\ \dot{y}_{pf}^{(n)} \\ \dot{z}_{pf}^{(n)} \end{bmatrix} + \dot{B}^T(R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + \left\{R_1 + B^T(R_2 - R_1)\right\} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} \quad . \qquad (7\text{-}19)$$

7.2 Contributions to the effective forces in the deformational freedoms

In the datum state a particle on the rotating part has coordinates $\left(x_{pf}^{(p)}, y_{pf}^{(p)}, z_{pf}^{(p)}\right)$, relative to the axial point $a = 1$ of (7-1), and referred to the axes fixed in the part (*cf* equation (7-18)). Thus, from the axisymmetric properties of the part, we have

$$\sum^{(p)} \delta m y_{pf}^{(p)} = \sum^{(p)} \delta m z_{pf}^{(p)} = 0 \qquad (7\text{-}20)$$

$$\sum^{(p)} \delta m {y_{pf}^{(p)}}^2 = \sum^{(p)} \delta m {z_{pf}^{(p)}}^2 \qquad (7\text{-}21)$$

$$\sum^{(p)} \delta m x_{pf}^{(p)} y_{pf}^{(p)} = \sum^{(p)} \delta m y_{pf}^{(p)} z_{pf}^{(p)} = \sum^{(p)} \delta m z_{pf}^{(p)} x_{pf}^{(p)} = 0 \qquad (7\text{-}22)$$

and hence (*cf* equation (2-4))

$$\Pi_f^T \dot{R}_{p_p t}^T \sum^{(p)} \delta m \begin{bmatrix} x_{pf}^{(p)} \\ y_{pf}^{(p)} \\ z_{pf}^{(p)} \end{bmatrix} = \sum^{(p)} \delta m \begin{bmatrix} \dot{x}_{pf}^{(n)} \\ \dot{y}_{pf}^{(n)} \\ \dot{z}_{pf}^{(n)} \end{bmatrix} = 0 \quad . \qquad (7\text{-}23)$$

Now, from equation (7-19)

$$\begin{bmatrix} \dfrac{\partial \dot{x}_n^{(n)}}{\partial q_1} & \dfrac{\partial \dot{y}_n^{(n)}}{\partial q_1} & \dfrac{\partial \dot{z}_n^{(n)}}{\partial q_1} \\ \vdots & \cdots\cdots & \cdots\cdots \\ \dfrac{\partial \dot{x}_n^{(n)}}{\partial q_n} & \cdots\cdots & \cdots\cdots \end{bmatrix} = (R_2 - R_1)^T \dot{B} \qquad (7\text{-}24)$$

and

$$\begin{bmatrix} \dfrac{\partial \dot{x}_n^{(n)}}{\partial \dot{q}_1} & \dfrac{\partial \dot{y}_n^{(n)}}{\partial \dot{q}_1} & \dfrac{\partial \dot{z}_n^{(n)}}{\partial \dot{q}_1} \\ \vdots & \cdots\cdots & \cdots\cdots \\ \dfrac{\partial \dot{x}_n^{(n)}}{\partial \dot{q}_n} & \cdots\cdots & \cdots\cdots \end{bmatrix} = R_1^T + (R_2 - R_1)^T B \quad . \qquad (7\text{-}25)$$

Also from equation (7-17)

$$\begin{bmatrix} \dfrac{\partial x_n^{(n)}}{\partial q_1} & \dfrac{\partial y_n^{(n)}}{\partial q_1} & \dfrac{\partial z_n^{(n)}}{\partial q_1} \\ \vdots & \cdots\cdots\cdots & \\ \dfrac{\partial x_n^{(n)}}{\partial q_n} & \cdots\cdots\cdots & \end{bmatrix} = R_1^T + (R_2 - R_1)^T B \tag{7-26}$$

and so using these equations along with (7-17), (7-19), (3-1) and (3-2), we find that the contributions to the effective forces, the left-hand side of Lagrange's equation (equation (4-1)), from the rotating part of the engine, are given by (*cf* equations (4-2) to (4-5))

$$\begin{bmatrix} G_1 \\ \vdots \\ G_n \end{bmatrix} = 2 \sum^{(p)} \delta m \left\{ R_1^T + (R_2 - R_1)^T B \right\} A_{p(n)} \times$$

$$\times \left(\begin{bmatrix} \dot{x}_{pf}^{(n)} \\ \dot{y}_{pf}^{(n)} \\ \dot{z}_{pf}^{(n)} \end{bmatrix} + \dot{B}^T (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + \left\{ R_1 + B^T (R_2 - R_1) \right\} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right) \qquad \ldots\ldots \tag{7-27}$$

$$\begin{bmatrix} J_1 \\ \vdots \\ J_n \end{bmatrix} = \sum^{(p)} \delta m \left\{ R_1^T + (R_2 - R_1)^T B \right\} A_{\dot{p}(n)} \times$$

$$\times \left(\begin{bmatrix} x_{f1} \\ y_{f1} \\ z_{f1} \end{bmatrix} + \begin{bmatrix} x_{pf}^{(n)} \\ y_{pf}^{(n)} \\ z_{pf}^{(n)} \end{bmatrix} + \left\{ R_1 + B^T (R_2 - R_1) \right\} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right) \tag{7-28}$$

$$\begin{bmatrix} \dfrac{\partial V_0}{\partial q_1} \\ \vdots \\ \dfrac{\partial V_0}{\partial q_n} \end{bmatrix} = \sum^{(p)} \delta m \left\{ R_1^T + (R_2 - R_1)^T B \right\} \times$$

$$\times \left(A^2_{p(n)} \left\{ \begin{bmatrix} x_{f1} \\ y_{f1} \\ z_{f1} \end{bmatrix} + \begin{bmatrix} x^{(n)}_{pf} \\ y^{(n)}_{pf} \\ z^{(n)}_{pf} \end{bmatrix} + \left\{ R_1 + B^T (R_2 - R_1) \right\} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right\} + A_{p(n)} \begin{bmatrix} u^{(n)} \\ v^{(n)} \\ w^{(n)} \end{bmatrix} + \begin{bmatrix} \dot{u}_n \\ \dot{v}_n \\ \dot{w}_n \end{bmatrix} \right) \qquad (7\text{-}29)$$

$$\begin{bmatrix} \dfrac{\partial W}{\partial q_1} \\ \vdots \\ \dfrac{\partial W}{\partial q_n} \end{bmatrix} = \sum^{(p)} \delta m (R_2 - R_1)^T \dot{B} \left\{ \begin{bmatrix} x^{(n)}_{pf} \\ y^{(n)}_{pf} \\ z^{(n)}_{pf} \end{bmatrix} + \dot{B}^T (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + \left(R_1 + B^T \{R_2 - R_1\} \right) \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} \right\}$$

$$\ldots\ldots \quad (7\text{-}30)$$

$$\begin{bmatrix} \dfrac{\partial W}{\partial \dot{q}_1} \\ \vdots \\ \dfrac{\partial W}{\partial \dot{q}_n} \end{bmatrix} = \sum^{(p)} \delta m \left\{ R_1^T + (R_2 - R_1)^T B \right\} \times$$

$$\times \left(\begin{bmatrix} \dot{x}^{(n)}_{pf} \\ \dot{y}^{(n)}_{pf} \\ \dot{z}^{(n)}_{pf} \end{bmatrix} + \dot{B}^T (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + \left\{ R_1 + B^T (R_2 - R_1) \right\} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} \right) .$$

$$\ldots\ldots \quad (7\text{-}31)$$

Now it is clear from the symmetry of the rotating part that none of the above functions (equations (7-27) to (7-31)) can depend on the specific value of $p_p t$ - it doesn't matter how far round the part has got in its rotation; and so the value of any term which does not involve a derivative of B or $\left\{x_{pf}^{(n)}\ y_{pf}^{(n)}\ z_{pf}^{(n)}\right\}$ will be independent of p_p. This can be demonstrated for example for such terms as

$$\left\{\begin{array}{l} \sum^{(p)} \delta m B \\ \sum^{(p)} \delta m B A_{x_{pf}^{(n)}} \\ \sum^{(p)} \delta m B B^T \\ \sum^{(p)} \delta m B A_{\xi} B^T \qquad \text{etc} \end{array}\right. \tag{7-32}$$

where $\{\xi\ \eta\ \zeta\}$ is any vector which is constant, with respect to location, over the part.

It follows that

$$\sum^{(p)} \delta m \dot{B} = 0 \tag{7-33}$$

$$\sum^{(p)} \delta m(\dot{B}B^T + B\dot{B}^T) = 0 \quad . \tag{7-34}$$

Thus, using the relationships (3-2) and (7-23)

$$\begin{bmatrix} G_1 \\ \vdots \\ G_n \end{bmatrix} - \begin{bmatrix} G_1 \\ \vdots \\ G_n \end{bmatrix}_{p_p=0} = 2(R_2 - R_1)^T \times$$

$$\times \left\{ \left(\sum^{(p)} \delta m B A_{p^{(n)}} \begin{bmatrix} \dot{x}_{pf}^{(n)} \\ \dot{y}_{pf}^{(n)} \\ \dot{z}_{pf}^{(n)} \end{bmatrix} \right) + \left(\sum^{(p)} \delta m B A_{p^{(n)}} \dot{B}^T \right) (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right\}$$

...... (7-35)

$$\begin{bmatrix} J_1 \\ \vdots \\ J_n \end{bmatrix} - \begin{bmatrix} J_1 \\ \vdots \\ J_n \end{bmatrix}_{p_p=0} = 0 \qquad (7\text{-}36)$$

$$\begin{bmatrix} \frac{\partial V_0}{\partial q_1} \\ \vdots \\ \frac{\partial V_0}{\partial q_n} \end{bmatrix} - \begin{bmatrix} \frac{\partial V_0}{\partial q_1} \\ \vdots \\ \frac{\partial V_0}{\partial q_n} \end{bmatrix}_{p_p=0} = 0 \qquad (7\text{-}37)$$

$$\frac{d}{dt}\begin{bmatrix} \frac{\partial W}{\partial \dot{q}_1} \\ \vdots \\ \frac{\partial W}{\partial \dot{q}_n} \end{bmatrix} - \left(\frac{d}{dt}\begin{bmatrix} \frac{\partial W}{\partial \dot{q}_1} \\ \vdots \\ \frac{\partial W}{\partial \dot{q}_n} \end{bmatrix}\right)_{p_p=0} = \frac{d}{dt}\left(\begin{bmatrix} \frac{\partial W}{\partial \dot{q}_1} \\ \vdots \\ \frac{\partial W}{\partial \dot{q}_n} \end{bmatrix} - \begin{bmatrix} \frac{\partial W}{\partial \dot{q}_1} \\ \vdots \\ \frac{\partial W}{\partial \dot{q}_n} \end{bmatrix}_{p_p=0}\right)$$

$$= \frac{d}{dt}\left((R_2 - R_1)^T \sum^{(p)} \delta m B\left\{\begin{bmatrix} \dot{x}^{(n)}_{pf} \\ \dot{y}^{(n)}_{pf} \\ \dot{z}^{(n)}_{pf} \end{bmatrix} + \dot{B}^T (R_2 - R_1)\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}\right\}\right)$$

$$= \begin{bmatrix} \frac{\partial W}{\partial q_1} \\ \vdots \\ \frac{\partial W}{\partial q_n} \end{bmatrix} - \begin{bmatrix} \frac{\partial W}{\partial q_1} \\ \vdots \\ \frac{\partial W}{\partial q_n} \end{bmatrix}_{p_p=0} + (R_2 - R_1)^T \sum^{(p)} \delta m \left\{ B \begin{bmatrix} \ddot{x}^{(n)}_{pf} \\ \ddot{y}^{(n)}_{pf} \\ \ddot{z}^{(n)}_{pf} \end{bmatrix} + B\ddot{B}^T (R_2 - R_1)\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}\right.$$

$$\left. + (B\dot{B}^T - \dot{B}B^T)(R_2 - R_1)\begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}\right\} \qquad \ldots\ldots\ (7\text{-}38)$$

and so the additional contribution to the effective forces due to the fact that part of the engine is rotating is

$$\begin{bmatrix} \dfrac{\partial V_0}{\partial q_1} + J_1 + G_1 + \dfrac{d}{dt}\left(\dfrac{\partial W}{\partial \dot{q}_1}\right) - \dfrac{\partial W}{\partial q_1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial V_0}{\partial q_n} + J_n + G_n + \dfrac{d}{dt}\left(\dfrac{\partial W}{\partial \dot{q}_n}\right) - \dfrac{\partial W}{\partial q_n} \end{bmatrix} - [\text{ditto}]_{p_p=0}$$

$$= (R_2 - R_1)^T \Bigg[-2\left(\sum^{(p)} \delta m B A_{\dot{x}^{(n)}_{pf}}\right)\begin{bmatrix} p^{(n)} \\ q^{(n)} \\ r^{(n)} \end{bmatrix} + 2\left(\sum^{(p)} \delta m B A_{p^{(n)}} \dot{B}^T\right)(R_2 - R_1)\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$+ \left(\sum^{(p)} \delta m B \begin{bmatrix} \ddot{x}^{(n)}_{pf} \\ \ddot{y}^{(n)}_{pf} \\ \ddot{z}^{(n)}_{pf} \end{bmatrix}\right) + \left(\sum^{(p)} \delta m B \ddot{B}^T\right)(R_2 - R_1)\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$+ \left(\sum^{(p)} \delta m \{B\dot{B}^T - \dot{B}B^T\}\right)(R_2 - R_1)\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \Bigg] \quad . \tag{7-39}$$

Now, using (7-16) and (7-18)

$$B\begin{bmatrix} x^{(n)}_{pf} \\ y^{(n)}_{pf} \\ z^{(n)}_{pf} \end{bmatrix} = -C\Pi_f A_{x^{(n)}_{pf}} \begin{bmatrix} x^{(n)}_{pf} \\ y^{(n)}_{pf} \\ z^{(n)}_{pf} \end{bmatrix} = 0 \tag{7-40}$$

$$B\begin{bmatrix} \ddot{x}^{(n)}_{pf} \\ \ddot{y}^{(n)}_{pf} \\ \ddot{z}^{(n)}_{pf} \end{bmatrix} = p_p^2 C R^T_{p_{pt}} A_{x^{(p)}_{pf}} \operatorname{diag}\{0 \quad 1 \quad 1\}\begin{bmatrix} x^{(p)}_{pf} \\ y^{(p)}_{pf} \\ z^{(p)}_{pf} \end{bmatrix}$$

$$= p_p^2 C R^T_{p_p t}\begin{bmatrix} 0 \\ -x^{(p)}_{pf} z^{(p)}_{pf} \\ x^{(p)}_{pf} y^{(p)}_{pf} \end{bmatrix} \tag{7-41}$$

since

$$R_{p_p t}\ddot{R}^T_{p_p t} = -p_p^2 \,\mathrm{diag}\{0\ 1\ 1\} \tag{7-42}$$

and so from (7-22)

$$\sum^{(p)} \delta m B \begin{bmatrix} \ddot{x}^{(n)}_{pf} \\ \ddot{y}^{(n)}_{pf} \\ \ddot{z}^{(n)}_{pf} \end{bmatrix} = 0 \quad . \tag{7-43}$$

Also

$$A_{x^{(n)}_{pf}} A_{\dot{x}^{(n)}_{pf}} = \Pi^T_f R^T_{p_p t}\left\{A_{x^{(p)}_{pf}} R_{p_p t}\dot{R}^T_{p_p t} A_{x^{(n)}_{pf}} + A^2_{x^{(p)}_{pf}} \dot{R}_{p_p t} R^T_{p_p t}\right\} R_{p_p t}\Pi_f$$

$$= \Pi^T_f R^T_{p_p t}\left\{R_{p_p t}\dot{R}^T_{p_p t} \begin{bmatrix} x^{(p)}_{pf} \\ y^{(p)}_{pf} \\ z^{(p)}_{pf} \end{bmatrix} \begin{bmatrix} x^{(p)}_{pf} & y^{(p)}_{pf} & z^{(p)}_{pf} \end{bmatrix}\right\} R_{p_p t}\Pi_f \tag{7-44}$$

since

$$R_{p_p t}\dot{R}^T_{p_p t} = p_p \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = -\dot{R}_{p_p t} R^T_{p_p t}$$

$$= \dot{R}^T_{p_p t} R_{p_p t} = -R^T_{p_p t}\dot{R}_{p_p t} \tag{7-45}$$

and so, using (7-22) and (7-21) along with these last two equations

$$\sum^{(p)} \delta m B A_{\dot{x}^{(n)}_{pf}} = -C\,\Pi_f \sum^{(p)} \delta m A_{x^{(n)}_{pf}} A_{\dot{x}^{(n)}_{pf}}$$

$$= -p_p\left(\sum^{(p)} \delta m {y^{(p)}_{pf}}^2\right) C \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \Pi_f \quad . \tag{7-46}$$

This relationship can be alternatively written as (see the Appendix, equation (A-23))

$$\sum^{(p)} \delta m B A_{\dot{x}_{pf}^{(n)}} = \frac{-p_p\left(\sum^{(p)} \delta m {y_{pf}^{(p)}}^2\right)}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{3}{2}}} A_{x_{pf}}^2 \quad . \tag{7-47}$$

Similarly

$$\sum^{(p)} \delta m B\dot{B}^T = \left(\sum^{(p)} \delta m B A_{\dot{x}_{pf}^{(n)}}\right) \Pi_f^T C^T$$

$$= \frac{-p_p\left(\sum^{(p)} \delta m {y_{pf}^{(p)}}^2\right)}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{3}{2}}} A_{x_{pf}}^2 \Pi_f^T C^T \tag{7-48}$$

and so, from (7-34)

$$\sum^{(p)} \delta m (B\dot{B}^T - \dot{B}B^T) = \frac{-2p_p\left(\sum^{(p)} \delta m {y_{pf}^{(p)}}^2\right)}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{3}{2}}} A_{x_{pf}}^2 \Pi_f^T C^T \quad . \tag{7-49}$$

Now it follows from equations (7-4), (7-10) and (7-16) that*

$$-C\Pi_f A_{x_{pf}} = I - \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \begin{bmatrix} x_{pf} & y_{pf} & z_{pf} \end{bmatrix} \tag{7-50}$$

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* We do not require the vector $\{\alpha \;\beta\; \gamma\}$. However, it can be shown that

$$-C\Pi_f A_{x_{pf}} = \frac{-1}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)} A_{x_{pf}}^2 \tag{7-50a}$$

and so

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \frac{1}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)} \begin{bmatrix} x_{pf} \\ y_{pf} \\ z_{pf} \end{bmatrix} \quad . \tag{7-50b}$$

and so (7-49) can be rewritten

$$\sum^{(p)} \delta m(B\dot{B}^T - \dot{B}B) = \frac{-2p_p\left(\sum^{(p)} \delta m y_{pf}^{(p)^2}\right)}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{3}{2}}} A_{x_{pf}} \quad . \tag{7-51}$$

Furthermore, since

$$R_{p_pt}\dot{R}_{p_pt} = -\dot{R}_{p_pt}R^T_{p_pt} = p_p\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{7-52}$$

and

$$\dot{R}_{p_pt}\dot{R}^T_{p_pt} = p_p^2 \operatorname{diag}\{0 \quad 1 \quad 1\} \tag{7-53}$$

we have, from (7-16) and (7-19)

$$\begin{aligned}
\dot{B}\dot{B}^T &= -\mathcal{C}\left(R^T_{p_pt}A_{x_{pf}^{(p)}}\dot{R}_{p_pt} + \dot{R}^T_{p_pt}A_{x_{pf}^{(p)}}R_{p_pt}\right)^2\mathcal{C}^T \\
&= -\mathcal{C}R^T_{p_pt}\left(A_{x_{pf}^{(p)}}\dot{R}_{p_pt}\dot{R}^T_{p_pt}A_{x_{pf}^{(p)}} + \left\{A_{x_{pf}^{(p)}}\dot{R}_{p_pt}R^T_{p_pt}\right\}^2\right. \\
&\qquad \left. + \left\{R_{p_pt}\dot{R}_{p_pt}A_{x_{pf}^{(p)}}\right\}^2 + R_{p_pt}\dot{R}^T_{p_pt}A^2_{x_{pf}^{(p)}}\dot{R}_{p_pt}R^T_{p_pt}\right)R_{p_pt}\mathcal{C}^T \\
&= -p_p^2\mathcal{C}R^T_{p_pt}\begin{bmatrix} -\left(y_{pf}^{(p)^2} + z_{pf}^{(p)^2}\right) & 0 & 0 \\ 0 & -y_{pf}^{(p)^2} & -y_{pf}^{(p)}z_{pf}^{(p)} \\ 0 & -y_{pf}^{(p)}z_{pf}^{(p)} & -z_{pf}^{(p)^2} \end{bmatrix}
\end{aligned} \tag{7-54}$$

and so (*cf* equations (7-21) and (7-22))

$$\sum^{(p)} \delta m\dot{B}\dot{B}^T = p_p^2\left(\sum^{(p)} \delta m y_{pf}^{(p)^2}\right)\mathcal{C} \operatorname{diag}\{2 \quad 1 \quad 1\}\mathcal{C}^T \quad . \tag{7-55}$$

Now from (7-48), differentiating

$$\sum^{(p)} \delta m \mathcal{B}\ddot{\mathcal{B}}^{T} = -\sum^{(p)} \delta m \dot{\mathcal{B}}\dot{\mathcal{B}}^{T} \tag{7-56}$$

and it can be shown that (see (7-7) and (7-15))

$$\mathcal{C}\Pi_{f} = \frac{1}{\left(x_{pf}^{2}+y_{pf}^{2}+z_{pf}^{2}\right)}\left\{A_{x_{pf}} + \frac{z_{pf}}{x_{pf}^{2}+y_{pf}^{2}}\begin{bmatrix} y_{pf} \\ -x_{pf} \\ 0 \end{bmatrix}\begin{bmatrix} x_{pf} & y_{pf} & z_{pf} \end{bmatrix}\right\} \tag{7-57}$$

and so

$$\mathcal{C}\mathcal{C}^{T} = \frac{1}{\left(x_{pf}^{2}+y_{pf}^{2}+z_{pf}^{2}\right)^{2}}\left\{-A_{x_{pf}}^{2} + \frac{z_{pf}^{2}\left(x_{pf}^{2}+y_{pf}^{2}+z_{pf}^{2}\right)}{\left(x_{pf}^{2}+y_{pf}^{2}\right)^{2}}\begin{bmatrix} y_{pf} \\ -x_{pf} \\ 0 \end{bmatrix}\begin{bmatrix} y_{pf} & -x_{pf} & 0 \end{bmatrix}\right\}.$$

...... (7-58)

Also

$$\mathcal{C}\,\mathrm{diag}\{1 \quad 0 \quad 0\}\mathcal{C}^{T} = \frac{z_{pf}^{2}}{\left(x_{pf}^{2}+y_{pf}^{2}+z_{pf}^{2}\right)\left(x_{pf}^{2}+y_{pf}^{2}\right)^{2}}\begin{bmatrix} y_{pf} \\ -x_{pf} \\ 0 \end{bmatrix}\begin{bmatrix} y_{pf} & -x_{pf} & 0 \end{bmatrix} .$$

...... (7-59)

Thus, the last five equations combine to give

$$\sum^{(p)} \delta m \mathcal{B}\ddot{\mathcal{B}}^{T} =$$

$$p_{p}^{2}\left[\frac{\left(\sum^{(p)} \delta m {y_{pf}^{(p)}}^{2}\right)}{\left(x_{pf}^{2}+y_{pf}^{2}+z_{pf}^{2}\right)^{2}}\left\{A_{x_{pf}}^{2} - \frac{2z_{pf}^{2}\left(x_{pf}^{2}+y_{pf}^{2}+z_{pf}^{2}\right)}{\left(x_{pf}^{2}+y_{pf}^{2}\right)^{2}}\begin{bmatrix} y_{pf} \\ -x_{pf} \\ 0 \end{bmatrix}\begin{bmatrix} y_{pf} & -x_{pf} & 0 \end{bmatrix}\right\}\right] .$$

...... (7-60)

The remaining summation, $\sum^{(p)} \delta m \mathcal{B} A_{p^{(n)}} \dot{\mathcal{B}}^T$, required in equation (7-39), is perhaps the most troublesome.

In the datum motion the velocity of a particle on the rotating part relative to the axis of rotation and referred to the axes 'fixed' in the rotating part, is by the usual formula

$$\begin{bmatrix} u^{(p)}_{pf} \\ v^{(p)}_{pf} \\ w^{(p)}_{pf} \end{bmatrix} = \begin{bmatrix} \dot{x}^{(p)}_{pf} \\ \dot{y}^{(p)}_{pf} \\ \dot{z}^{(p)}_{pf} \end{bmatrix} + A_{\left(\begin{bmatrix} p_p \\ 0 \\ 0 \end{bmatrix}\right)} \begin{bmatrix} x^{(p)}_{pf} \\ y^{(p)}_{pf} \\ z^{(p)}_{pf} \end{bmatrix} = p_p \begin{bmatrix} 0 \\ -z^{(p)}_{pf} \\ y^{(p)}_{pf} \end{bmatrix} \tag{7-61}$$

since the $x^{(p)}_{pf}$ are constant. Consequently, transforming to the no-deformation-body-fixed axes, we have

$$\begin{bmatrix} \dot{x}^{(n)}_{pf} \\ \dot{y}^{(n)}_{pf} \\ \dot{z}^{(n)}_{pf} \end{bmatrix} = \Pi^T_f R^T_{p_p t} \begin{bmatrix} u^{(p)}_{pf} \\ v^{(p)}_{pf} \\ w^{(p)}_{pf} \end{bmatrix} = p_p \Pi^T_f R^T_{p_p t} \begin{bmatrix} 0 \\ -z^{(p)}_{pf} \\ y^{(p)}_{pf} \end{bmatrix} . \tag{7-62}$$

Using this equation, along with (7-18) and (7-16), we have

$$\mathcal{B} A_{p^{(n)}} \dot{\mathcal{B}}^T = -\mathcal{C} \Pi_f \left(A_{x^{(n)}_{pf}} A_{p^{(n)}} A_{\dot{x}^{(n)}_{pf}} \right) \Pi^T_f \mathcal{C}^T$$

$$= -\mathcal{C} R^T_{p_p t} A_{x^{(p)}_{pf}} A_{\left(R_{p_p t} \Pi \begin{bmatrix} p^{(n)} \\ q^{(n)} \\ r^{(n)} \end{bmatrix} \right)} A_{u^{(p)}_{pf}} R_{p_p t} \mathcal{C}^T . \tag{7-63}$$

Now, as mentioned earlier, the sum of this term ($\times$ δm) over the rotating part must be independent of the value of $p_p t$, and so

$$\sum^{(p)} \delta m \mathcal{B} A_{p^{(n)}} \dot{\mathcal{B}}^T = -\mathcal{C} \left(\sum^{(p)} \delta m A_{x^{(p)}_{pf}} A_{\left(\Pi_f \begin{bmatrix} p^{(n)} \\ q^{(n)} \\ r^{(n)} \end{bmatrix} \right)} A_{u^{(p)}_{pf}} \right) \mathcal{C}^T . \tag{7-64}$$

If we write

$$\Pi_f \begin{bmatrix} p^{(n)} \\ q^{(n)} \\ r^{(n)} \end{bmatrix} = \begin{bmatrix} p^{(p0)} \\ q^{(p0)} \\ r^{(p0)} \end{bmatrix} \tag{7-65}$$

then, using (7-61)

$$\begin{aligned} {}^A x_{pf}^{(p)}\, {}^A p^{(p0)}\, {}^A u_{pf}^{(p)} &= p_p\, {}^A x_{pf}^{(p)} \left\{ \begin{bmatrix} 0 \\ -z_{pf}^{(p)} \\ y_{pf}^{(p)} \end{bmatrix} \begin{bmatrix} p^{(p0)} & q^{(p0)} & r^{(p0)} \end{bmatrix} - \left(r^{(p0)} y_{pf}^{(p)} - q^{(p0)} z_{pf}^{(p)} \right) I \right\} \\ &= p_p \left\{ \begin{bmatrix} y_{pf}^{(p)2} + z_{pf}^{(p)2} \\ -x_{pf}^{(p)} y_{pf}^{(p)} \\ -x_{pf}^{(p)} z_{pf}^{(p)} \end{bmatrix} \begin{bmatrix} p^{(p0)} & q^{(p0)} & r^{(p0)} \end{bmatrix} + q^{(p0)} \begin{bmatrix} 0 & -z_{pf}^{(p)2} & y_{pf}^{(p)} z_{pf}^{(p)} \\ z_{pf}^{(p)2} & 0 & -x_{pf}^{(p)} z_{pf}^{(p)} \\ -y_{pf}^{(p)} z_{pf}^{(p)} & x_{pf}^{(2)} z_{pf}^{(2)} & 0 \end{bmatrix} \right. \\ &\qquad \left. - r^{(p0)} \begin{bmatrix} 0 & -y_{pf}^{(p)} z_{pf}^{(p)} & y_{pf}^{(p)2} \\ y_{pf}^{(p)} z_{pf}^{(p)} & 0 & -x_{pf}^{(p)} y_{pf}^{(p)} \\ -y_{pf}^{(p)2} & x_{pf}^{(p)} y_{pf}^{(p)} & 0 \end{bmatrix} \right\} . \end{aligned}$$

...... (7-66)

Thus, taking account of (7-21) and (7-22), as well as (7-65),

$$\begin{aligned} \sum\nolimits^{(p)} \delta m\, {}^A x_{pf}^{(p)}\, {}^A p^{(p0)}\, {}^A u_{pf}^{(p)} &= p_p \left(\sum\nolimits^{(p)} \delta m y_{pf}^{(p)2} \right) \begin{bmatrix} 2p^{(p0)} & q^{(p0)} & r^{(p0)} \\ q^{(p0)} & 0 & 0 \\ r^{(p0)} & 0 & 0 \end{bmatrix} \\ &= p_p \left(\sum\nolimits^{(p)} \delta m y_{pf}^{(p)2} \right) \left\{ \Pi_f \begin{bmatrix} p^{(n)} \\ q^{(n)} \\ r^{(n)} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} p^{(n)} & q^{(n)} & r^{(n)} \end{bmatrix} \Pi_f^T \right\} \end{aligned}$$

...... (7-67)

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and so

$$\sum\nolimits^{(p)} \delta m \mathcal{B} A_{p^{(n)}} \dot{\mathcal{B}}^{T} =$$

$$- p_p \left(\sum\nolimits^{(p)} \delta m {y_{pf}^{(p)}}^2 \right) \left\{ \mathcal{C}\Pi_f \begin{bmatrix} p^{(n)} \\ q^{(n)} \\ r^{(n)} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathcal{C}^T + \mathcal{C} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} p^{(n)} & q^{(n)} & r^{(n)} \end{bmatrix} \Pi_f^T \mathcal{C}^T \right\} \quad . \qquad \ldots\ldots \; (7\text{-}68)$$

This equation can be written in an alternative form, using the expression for $\mathcal{C}\Pi_f$ given in equation (7-57), and that for $\mathcal{C}$ (equation (7-15)), viz

$$\mathcal{C}\Pi_f \begin{bmatrix} p^{(n)} \\ q^{(n)} \\ r^{(n)} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathcal{C}^T$$

$$= \frac{z_{pf}}{\left(x_{pf}^2 + y_{pf}^2\right)\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{1}{2}}} \left\{ {}^{A}x_{pf} + \frac{z_{pf}}{x_{pf}^2 + y_{pf}^2} \begin{bmatrix} y_{pf} \\ -x_{pf} \\ 0 \end{bmatrix} \begin{bmatrix} x_{pf} & y_{pf} & z_{pf} \end{bmatrix} \right\} \begin{bmatrix} p^{(n)} \\ q^{(n)} \\ r^{(n)} \end{bmatrix} \begin{bmatrix} y_{pf} & -x_{pf} & 0 \end{bmatrix} \qquad \ldots\ldots \; (7\text{-}69)$$

and so

$$\sum\nolimits^{(p)} \delta m \mathcal{B} A_{p^{(n)}} \dot{\mathcal{B}}^{T}$$

$$= \frac{-2p_p \left(\sum\nolimits^{(p)} \delta m {y_{pf}^{(p)}}^2 \right) z_{pf}^2}{\left(x_{pf}^2 + y_{pf}^2\right)^2 \left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{1}{2}}} \begin{bmatrix} y_{pf} \\ -x_{pf} \\ 0 \end{bmatrix} \begin{bmatrix} x_{pf} & y_{pf} & z_{pf} \end{bmatrix} \begin{bmatrix} p^{(n)} \\ q^{(n)} \\ r^{(n)} \end{bmatrix} \begin{bmatrix} y_{pf} & -x_{pf} & 0 \end{bmatrix} \quad . \qquad \ldots\ldots \; (7\text{-}70)$$

The additional contribution to the effective forces due to the fact that part of the engine is rotating is therefore, from (7-39), using the results (7-43) and either (7-46), (7-68), (7-55), (7-56) and (7-49), or (7-47), (7-70), (7-60), (7-51) and (3-2), accordingly as we wish to express the results in terms of the matrix C or not,

$$\begin{bmatrix} \frac{\partial V_0}{\partial q_1} + J_1 + G_1 + \frac{d}{dt}\left(\frac{\partial W}{\partial \dot q_1}\right) - \frac{\partial W}{\partial q_1} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \frac{\partial V_0}{\partial q_n} + J_n + G_n + \frac{d}{dt}\left(\frac{\partial W}{\partial \dot q_n}\right) - \frac{\partial W}{\partial q_n} \end{bmatrix} - \begin{bmatrix}\text{Ditto}\end{bmatrix}_{p_p=0}$$

$$= \left(\sum\nolimits^{(p)} \delta m y_{pf}^{(p)^2}\right)(R_2 - R_1)^T \left\{\left(2C\begin{bmatrix}0 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{bmatrix}\Pi_f\begin{bmatrix}p^{(n)}\\q^{(n)}\\r^{(n)}\end{bmatrix} - \frac{2}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{3}{2}}} A_{x_{pf}}(R_2 - R_1)\begin{bmatrix}\dot q_1\\ \vdots \\ \dot q_n\end{bmatrix}\right.\right.$$

$$\left.\left. - 2\left\{C\Pi_f\begin{bmatrix}p^{(n)}\\q^{(n)}\\r^{(n)}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\end{bmatrix}C^T + C\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}p^{(n)} & q^{(n)} & r^{(n)}\end{bmatrix}\Pi_f^T C^T\right\}(R_2 - R_1)\begin{bmatrix}q_1\\ \vdots \\ q_n\end{bmatrix}\right) p_p - \left(C \operatorname{diag}\{2 \quad 1 \quad 1\} C^T (R_2 - R_1)\begin{bmatrix}q_1\\ \vdots \\ q_n\end{bmatrix}\right) p_p^2\right\}$$

$$= \left(\sum\nolimits^{(p)} \delta m y_{pf}^{(p)^2}\right)\left\{\frac{2p_p(R_2 - R_1)^T}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{3}{2}}}\left(A_{x_{pf}}^2\begin{bmatrix}\dot\phi\\ \dot\theta\\ \dot\psi\end{bmatrix} - A_{x_{pf}}(R_2 - R_1)\begin{bmatrix}\dot q_1\\ \vdots \\ \dot q_n\end{bmatrix} + \cdots\right) \frac{-p_p^2(R_2 - R_1)^T}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^2}\left(A_{x_{pf}}^2 - \frac{2z_{pf}^2\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)}{\left(x_{pf}^2 + y_{pf}^2\right)^2}\begin{bmatrix}y_{pf}\\ -x_{pf}\\ 0\end{bmatrix}\begin{bmatrix}y_{pf} & -x_{pf} & 0\end{bmatrix}\right)(R_2 - R_1)\begin{bmatrix}q_1\\ \vdots \\ q_n\end{bmatrix}\right\}$$

...... (7-71)

The latter form has only been given explicitly as far as the first order terms in the generalised coordinates (in particular the contribution from (7-70) does not appear), for it is thought the former form will be the more convenient if one wishes to venture further. Thus the second order terms in (7-71) are (*cf* (3-2), (3-3) and (7-88) or (A-23))

$$- \left(\sum\nolimits^{(p)} \delta m y_{pf}^{(p)^2}\right)(R_2 - R_1)^T 2C\left\{\begin{bmatrix}0 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{bmatrix}\Pi_f J_\phi\begin{bmatrix}\dot\phi\\ \dot\theta\\ \dot\psi\end{bmatrix} + \Pi_f\begin{bmatrix}\dot\phi\\ \dot\theta\\ \dot\psi\end{bmatrix}\begin{bmatrix}1 & 0 & 0\end{bmatrix}C^T(R_2 - R_1)\begin{bmatrix}q_1\\ \vdots \\ q_n\end{bmatrix}p_p\right\}$$

$$= -2\left(\sum\nolimits^{(p)} \delta m y_{pf}^{(p)^2}\right)p_p\left\{\frac{A_{x_{pf}}^2 J_\phi}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{3}{2}}}\begin{bmatrix}\dot\phi\\ \dot\theta\\ \dot\psi\end{bmatrix} + C\Pi_f\begin{bmatrix}\dot\phi\\ \dot\theta\\ \dot\psi\end{bmatrix}\begin{bmatrix}1 & 0 & 0\end{bmatrix}C^T(R_2 - R_1)\begin{bmatrix}q_1\\ \vdots \\ q_n\end{bmatrix}\right\}.$$

.... (7-72)

7.3 Contributions to the effective forces in the body-freedom equations

The right hand sides of equations (4-17) and (4-18) are respectively[1,2]

$$\sum \delta m \left\{ \begin{bmatrix} \dot{u}_m^{(n)} \\ \dot{v}_m^{(n)} \\ \dot{w}_m^{(n)} \end{bmatrix} + A_{p}^{(n)} \begin{bmatrix} u_m^{(n)} \\ v_m^{(n)} \\ w_m^{(n)} \end{bmatrix} \right\} \tag{7-73}$$

and

$$\sum \delta m A_{x_n}^{(n)} \left\{ \begin{bmatrix} \dot{u}_m^{(n)} \\ \dot{v}_m^{(n)} \\ \dot{w}_m^{(n)} \end{bmatrix} + A_{p}^{(n)} \begin{bmatrix} u_m^{(n)} \\ v_m^{(n)} \\ w_m^{(n)} \end{bmatrix} \right\} \tag{7-74}$$

where the constituents are given by equations (3-6), (3-2) and (2-1). Now on the rotating part (2-1) becomes (7-17) and its derivative is (7-19). Thus we have, from (3-6),

$$\begin{bmatrix} u_m^{(n)} \\ v_m^{(n)} \\ w_m^{(n)} \end{bmatrix} = S\left\{ \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} + \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} \right\} + \begin{bmatrix} \dot{x}_{pf}^{(n)} \\ \dot{y}_{pf}^{(n)} \\ \dot{z}_{pf}^{(n)} \end{bmatrix} + \dot{\mathcal{B}}^T(R_2 - R_1)\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + \left\{R_1 + \mathcal{B}^T(R_2 - R_1)\right\}\begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$+ A_{p}^{(n)}\left(\begin{bmatrix} x_{f1} \\ y_{f1} \\ z_{f1} \end{bmatrix} + \begin{bmatrix} x_{pf}^{(n)} \\ y_{pf}^{(n)} \\ z_{pf}^{(n)} \end{bmatrix} + \left\{R_1 + \mathcal{B}^T(R_2 - R_1)\right\}\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right) .$$

.... (7-75)

Now, making use of the axisymmetric properties (equations (7-20) to (7-22)) of the rotating part it can be shown, after some manipulation, that

$$\sum^{(p)} \delta m A_{x_{pf}}^{(n)} \begin{bmatrix} \ddot{x}_{pf}^{(n)} \\ \ddot{y}_{pf}^{(n)} \\ \ddot{z}_{pf}^{(n)} \end{bmatrix} = 0 \tag{7-76}$$

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$$\sum^{(p)} \delta m A_{x_{pf}^{(n)}} \dot{B}^{T} = -\, p_p \left(\sum^{(p)} \delta m y_{pf}^{(p)2} \right) \Pi_f^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} C^T \tag{7-77}$$

$$\sum^{(p)} \delta m A_{x_{pf}^{(n)}} \ddot{B}^{T} = p_p^2 \left(\sum^{(p)} \delta m y_{pf}^{(p)2} \right) \Pi_f^T \,\mathrm{diag}\{2 \quad 1 \quad 1\} C^T \tag{7-78}$$

$$\sum^{(p)} \delta m A_{x_{pf}^{(n)}} A_{p^{(n)}} \begin{bmatrix} \dot{x}_{pf}^{(n)} \\ \dot{y}_{pf}^{(n)} \\ \dot{z}_{pf}^{(n)} \end{bmatrix} = p_p \left(\sum^{(p)} \delta m y_{pf}^{(p)2} \right) \Pi_f^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \Pi_f \begin{bmatrix} p^{(n)} \\ q^{(n)} \\ r^{(n)} \end{bmatrix} \tag{7-79}$$

$$\sum^{(p)} \delta m \left\{ A_{x_{pf}^{(n)}} A_{p^{(n)}} \dot{B}^T (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + A_{\left(B^T (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}\right)} A_{p^{(n)}} \begin{bmatrix} \dot{x}_{pf}^{(n)} \\ \dot{y}_{pf}^{(n)} \\ \dot{z}_{pf}^{(n)} \end{bmatrix} \right\}$$

$$= -\, p_p \left(\sum^{(p)} \delta m y_{pf}^{(p)2} \right) A_{p^{(n)}} \Pi_f^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} C^T (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \tag{7-80}$$

$$\sum^{(p)} \delta m A_{\left(B^T (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}\right)} A_{p^{(n)}} \dot{B}^T (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$= p_p \left(\sum^{(p)} \delta m y_{pf}^{(p)2} \right) \Pi_f^T A_{\left(C^T (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}\right)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} A_{\left(C^T (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}\right)} \Pi_f \begin{bmatrix} p^{(n)} \\ q^{(n)} \\ r^{(n)} \end{bmatrix}$$

(7-81)

$$\sum^{(p)} \delta m A_{\left(B^T(R_2-R_1)\begin{bmatrix}q_1\\ \vdots\\ q_n\end{bmatrix}\right)} \begin{bmatrix}\ddot{x}^{(n)}_{pf}\\ \ddot{y}^{(n)}_{pf}\\ \ddot{z}^{(n)}_{pf}\end{bmatrix}$$

$$= \; - p_p^2\left(\sum^{(p)} \delta m y^{(p)\,2}_{pf}\right)\Pi_f^T \operatorname{diag}\{2 \quad 1 \quad 1\}C^T(R_2-R_1)\begin{bmatrix}q_1\\ \vdots\\ q_n\end{bmatrix} \qquad (7\text{-}82)$$

$$\sum^{(p)} \delta m A_{\left(B^T(R_2-R_1)\begin{bmatrix}q_1\\ \vdots\\ q_n\end{bmatrix}\right)} \ddot{B}(R_2-R_1)\begin{bmatrix}q_1\\ \vdots\\ q_n\end{bmatrix} = 0 \qquad (7\text{-}83)$$

and

$$\sum^{(p)} \delta m A_{\left(B^T(R_2-R_1)\begin{bmatrix}q_1\\ \vdots\\ q_n\end{bmatrix}\right)} \dot{B}^T(R_2-R_1)\begin{bmatrix}\dot{q}_1\\ \vdots\\ \dot{q}_n\end{bmatrix}$$

$$= p_p\left(\sum^{(p)} \delta m y^{(p)\,2}_{pf}\right)\Pi_f^T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}q_1 & \cdots & q_n\end{bmatrix}(R_2-R_1)^T C + \begin{bmatrix}1 & 0 & 0\end{bmatrix}C^T(R_2-R_1)\begin{bmatrix}q_1\\ \vdots\\ q_n\end{bmatrix} I\right)C^T(R_2-R_1)\begin{bmatrix}\dot{q}_1\\ \vdots\\ \dot{q}_n\end{bmatrix}$$

$$\ldots\ldots \quad (7\text{-}84)$$

where the coefficient of the unit matrix I is a scalar.

Also from (7-33)

$$\sum \delta m \ddot{B}^T = 0 \quad . \qquad (7\text{-}85)$$

Using the above relationships, along with (7-23) and (7-33) we then find that

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$$\sum^{(p)} \delta m \left\{ \begin{bmatrix} \dot{u}_m^{(n)} \\ \dot{v}_m^{(n)} \\ \dot{w}_m^{(n)} \end{bmatrix} + A_{p(n)} \begin{bmatrix} u_m^{(n)} \\ v_m^{(n)} \\ w_m^{(n)} \end{bmatrix} \right\} - (\text{Ditto})_{p_p} = 0$$

$$= \sum^{(p)} \delta m \left\{ \begin{bmatrix} \ddot{x}_{pf}^{(n)} \\ \ddot{y}_{pf}^{(n)} \\ \ddot{z}_{pf}^{(n)} \end{bmatrix} + \ddot{B}^T (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + 2\dot{B}^T (R_2 - R_1) \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} \right.$$

$$\left. + 2A_{p(n)} \left(\begin{bmatrix} \dot{x}_{pf}^{(n)} \\ \dot{y}_{pf}^{(n)} \\ \dot{z}_{pf}^{(n)} \end{bmatrix} + \dot{B}^T (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right) \right\} = 0$$

...... (7-86)

and

$$\sum^{(p)} \delta m A_{x_n^{(n)}} \left\{ \begin{bmatrix} \dot{u}_m^{(n)} \\ \dot{v}_m^{(n)} \\ \dot{w}_m^{(n)} \end{bmatrix} + A_{p(n)} \begin{bmatrix} u_m^{(n)} \\ v_m^{(n)} \\ w_m^{(n)} \end{bmatrix} \right\} - (\text{Ditto})_{p_p} = 0$$

$$= \sum^{(p)} \delta m \left\{ A_{x_{f1}} + A_{x_{pf}^{(n)}} + A_{\left(R_1 \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right)} + A_{\left(B^T (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right)} \right\} \left\{ \begin{bmatrix} \dot{u}_m^{(n)} \\ \dot{v}_m^{(n)} \\ \dot{w}_m^{(n)} \end{bmatrix} + A_{p(n)} \begin{bmatrix} u_m^{(n)} \\ v_m^{(n)} \\ w_m^{(n)} \end{bmatrix} \right\}$$

$$= 2\left(\sum^{(p)} \delta m y_{pf}^{(p)\,2} \right) p_p \left\{ \Pi_f^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \left(C^T (R_2 - R_1) \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} + \Pi_f \begin{bmatrix} p^{(n)} \\ q^{(n)} \\ r^{(n)} \end{bmatrix} \right) - A_{p(n)} \Pi_f^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} C^T (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right.$$

$$+ \Pi_f^T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} (R_2 - R_1)^T C + \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} C^T (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} I \right) C^T (R_2 - R_1) \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\left. + \Pi_f^T A_{\left(C^T (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} A_{\left(C^T (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right)} \Pi_f \begin{bmatrix} p^{(n)} \\ q^{(n)} \\ r^{(n)} \end{bmatrix} \right\} \quad (7\text{-}87)$$

Some simplification of the above expression can be made since, from (A-23),

$$\Pi_f^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} C^T = \frac{A_{x_{pf}}^2}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{3}{2}}} \tag{7-88}$$

and also

$$\Pi_f^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \Pi_f = \frac{-A_{x_{pf}}}{\sqrt{x_{pf}^2 + y_{pf}^2 + z_{pf}^2}} \ . \tag{7-89}$$

Consequently, using (3-2) and (2-13),

$$\sum{}^{(p)} \delta m A_{x_n^{(n)}} \left\{ \begin{bmatrix} \dot{u}_m^{(n)} \\ \dot{v}_m^{(n)} \\ \dot{w}_m^{(n)} \end{bmatrix} + A_{\Gamma^{(n)}} \begin{bmatrix} u_m^{(n)} \\ v_m^{(n)} \\ w_m^{(n)} \end{bmatrix} \right\} - (\text{Ditto})_{p_p} = 0$$

$$= 2\left(\sum{}^{(p)} \delta m y_{pf}^{(p)\,2}\right)_{p_p} \left\{ \begin{bmatrix} \dfrac{-A_{x_{pf}}^2 (R_2 - R_1)}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{3}{2}}} & 0 & \dfrac{-A_{x_{pf}}}{\sqrt{x_{pf}^2 + y_{pf}^2 + z_{pf}^2}} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_{n+6} \end{bmatrix} \right.$$

$$+ \frac{A_{x_{pf}} J_\phi}{\sqrt{x_{pf}^2 + y_{pf}^2 + z_{pf}^2}} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} - \frac{A_{\dot{\phi}} A_{x_{pf}}^2 (R_2 - R_1)}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{3}{2}}} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$+ \Pi_f^T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} (R_2 - R_1)^T C + \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} C^T (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} I \right) C^T (R_2 - R_1) \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\left. + \text{ third and higher order terms} \right\} \ . \tag{7-90}$$

7.4 The 'genuine' propulsive forces

In Ref 2 it was assumed that the propulsive force acting on any particular particle had constant components in the direction of the body-fixed axes*. However, as indicated in Ref 4, it is only convenient to do this when the body-fixed axes are constrained to remain mutually at right angles during small perturbations. Body-fixed axes do not feature at all in the present development in any other respect and so it seems rather artificial to introduce them in a constrained form now. We have however (*cf* section 7.1) used axes which are virtually fixed in the rotating part (or parts) of the propulsive unit and these axes are such that they are always mutually perpendicular. It is therefore as good an assumption as that used previously[2] to say that the local propulsive force on the rotating part has constant components in the directions of these latter axes, while on the adjacent non-rotating part the components in the directions of any instantaneous position of these axes are constant. Let $\left(e^{(p)}_{pf},\ f^{(p)}_{pf},\ g^{(p)}_{pf}\right)$ be the components of the local propulsive force referred to the axes fixed (virtually) in the rotating part. Then our assumptions are

$$\begin{bmatrix} e^{(p)}_{pf} \\ f^{(p)}_{pf} \\ g^{(p)}_{pf} \end{bmatrix} = \text{constant} \qquad \text{on the rotating part} \qquad (7\text{-}91)$$

$$R^{T}_{P_p t} \begin{bmatrix} e^{(p)}_{pf} \\ f^{(p)}_{pf} \\ g^{(p)}_{pf} \end{bmatrix} = \text{constant} \qquad \text{on the adjacent non-rotating part.} \qquad (7\text{-}92)$$

The propulsive force on the non-rotating part, referred to the no-deformation-body-fixed axes is obtained by premultiplying the last expression by Π^T and so will not have constant components. We are able to avoid greater complication, such as that of Ref 4, because we again have some local constraint on the modes - the assumption of almost rigidity for the rotating part produced the conditions (7-4) and (7-10), though different from that of Ref 2; while in Ref 4 there was no constraint at all on the modal matrix function R .

---

* Body-fixed axes are axes which are fixed in a small material portion of the body.

The local propulsive force vector referred to the no-deformation-body-fixed axes is therefore (*cf* equation (7-8))

$$\begin{bmatrix} e_p^{(n)} \\ f_p^{(n)} \\ g_p^{(n)} \end{bmatrix} = \Pi^T R^T_{p_p t} \begin{bmatrix} e_{pf}^{(p)} \\ f_{pf}^{(p)} \\ g_{pf}^{(p)} \end{bmatrix} \approx \Pi_f^T R^T_{p_p t} \begin{bmatrix} e_{pf}^{(p)} \\ f_{pf}^{(p)} \\ g_{pf}^{(p)} \end{bmatrix} + \left\{ \left[\frac{\partial \Pi_f^T}{\partial x_{pf}} R^T_{p_p t} \begin{bmatrix} e_{pf}^{(p)} \\ f_{pf}^{(p)} \\ g_{pf}^{(p)} \end{bmatrix} \quad \frac{\partial \Pi_f^T}{\partial y_{pf}} R^T_{p_p t} \begin{bmatrix} e_{pf}^{(p)} \\ f_{pf}^{(p)} \\ g_{pf}^{(p)} \end{bmatrix} \quad \frac{\partial \Pi_f^T}{\partial z_{pf}} R^T_{p_p t} \begin{bmatrix} e_{pf}^{(p)} \\ f_{pf}^{(p)} \\ g_{pf}^{(p)} \end{bmatrix} \right] \right.$$

$$\left. + \frac{1}{2} \begin{bmatrix} [q_1 \cdots q_n](R_2 - R_1)^T P_1 \\ [q_1 \cdots q_n](R_2 - R_1)^T P_2 \\ [q_1 \cdots q_n](R_2 - R_1)^T P_3 \end{bmatrix} \right\} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \qquad (7\text{-}93)$$

where the matrices P_i are analogous to the B_i of equation (7-8); *ie* they are symmetric and such that

$$\left.\begin{array}{l} (11) \text{ element of } P_i = i\text{th element of } \begin{bmatrix} e_{pf}^{(p)} & f_{pf}^{(p)} & g_{pf}^{(p)} \end{bmatrix} R_{p_p t} \dfrac{\partial^2 \Pi_f}{\partial x_{pf}^2} \\ \\ (12) \text{ element of } P_i = i\text{th element of } \begin{bmatrix} e_{pf}^{(p)} & f_{pf}^{(p)} & g_{pf}^{(p)} \end{bmatrix} R_{p_p t} \dfrac{\partial^2 \Pi_f}{\partial x_{pf} \partial y_{pf}} \end{array}\right\}$$

$$\ldots \text{ etc} . \qquad (7\text{-}94)$$

If we define, with $e_{pf}^{(n)}$ the unperturbed value of $e_p^{(n)}$ (see equation (7-93)),

$$P = \begin{bmatrix} \begin{bmatrix} e_{pf}^{(n)} & f_{pf}^{(n)} & g_{pf}^{(n)} \end{bmatrix} \Pi_f^T \dfrac{\partial \Pi_f}{\partial x_{pf}} \\ \begin{bmatrix} e_{pf}^{(n)} & f_{pf}^{(n)} & g_{pf}^{(n)} \end{bmatrix} \Pi_f^T \dfrac{\partial \Pi_f}{\partial y_{pf}} \\ \begin{bmatrix} e_{pf}^{(n)} & f_{pf}^{(n)} & g_{pf}^{(n)} \end{bmatrix} \Pi_f^T \dfrac{\partial \Pi_f}{\partial z_{pf}} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} e_{pf}^{(p)} & f_{pf}^{(p)} & g_{pf}^{(p)} \end{bmatrix} R_{p_p t} \dfrac{\partial \Pi_f}{\partial x_{pf}} \\ \begin{bmatrix} e_{pf}^{(p)} & f_{pf}^{(p)} & g_{pf}^{(p)} \end{bmatrix} R_{p_p t} \dfrac{\partial \Pi_f}{\partial y_{pf}} \\ \begin{bmatrix} e_{pf}^{(p)} & f_{pf}^{(p)} & g_{pf}^{(p)} \end{bmatrix} R_{p_p t} \dfrac{\partial \Pi_f}{\partial z_{pf}} \end{bmatrix}, \qquad (7\text{-}95)$$

analogous to the expression for B (equation (7-14)); then, for any vector $\{\xi \;\eta\; \zeta\}$,

$$\begin{bmatrix} \frac{\partial P^T}{\partial x_{pf}} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} & \frac{\partial P^T}{\partial y_{pf}} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} & \frac{\partial P^T}{\partial z_{pf}} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \end{bmatrix} = \begin{bmatrix} [\xi \quad \eta \quad \zeta] P_1 \\ [\xi \quad \eta \quad \zeta] P_2 \\ [\xi \quad \eta \quad \zeta] P_3 \end{bmatrix} \tag{7-96}$$

and so (7-93) can be rewritten as

$$\begin{bmatrix} e_p^{(n)} \\ f_p^{(n)} \\ g_p^{(n)} \end{bmatrix} \approx \Pi_f^T R_{P_p t}^T \begin{bmatrix} e_{pf}^{(p)} \\ f_{pf}^{(p)} \\ g_{pf}^{(p)} \end{bmatrix} + P^T (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$+ \frac{1}{2} \begin{bmatrix} \frac{\partial P^T}{\partial x_{pf}} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} & \frac{\partial P^T}{\partial y_{pf}} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} & \frac{\partial P^T}{\partial z_{pf}} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \end{bmatrix} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

...... (7-97)

A further condition on the propulsive force components, which we will apply, is that, *in the unperturbed state, the total propulsive force due to each rotating part will have constant components referred to the no-deformation-body-fixed axes*, and so

$$\dot{R}_{P_p t}^T \sum^{(p)} \begin{bmatrix} e_{pf}^{(p)} \\ f_{pf}^{(p)} \\ g_{pf}^{(p)} \end{bmatrix} = 0 \quad . \tag{7-98}$$

The total region where the propulsive force acts, denoted by (p+), is the rotating part, plus a bit more, but over (p+) - (p) this condition is satisfied as a consequence of (7-92). From the form of $R_{P_p t}$ (*cf* equation (2-4)) we then have

$$R_{P_p t}^T \sum^{(p)} \begin{bmatrix} e_{pf}^{(p)} \\ f_{pf}^{(p)} \\ g_{pf}^{(p)} \end{bmatrix} = \begin{bmatrix} \sum^{(p)} e_{pf}^{(p)} \\ 0 \\ 0 \end{bmatrix} \tag{7-99}$$

and

$$\sum\nolimits^{(p)} f_{pf}^{(p)} = \sum\nolimits^{(p)} g_{pf}^{(p)} = 0 \quad . \tag{7-100}$$

Finally, we assume that the moment of the propulsive forces on the rotating part is about the axis of rotation and so

$$\sum\nolimits^{(p)} A_{x_{pf}^{(p)}} \begin{bmatrix} e_{pf}^{(p)} \\ f_{pf}^{(p)} \\ g_{pf}^{(p)} \end{bmatrix} = \begin{bmatrix} \sum\nolimits^{(p)} \left\{ g_{pf}^{(p)} y_{pf}^{(p)} - f_{pf}^{(p)} z_{pf}^{(p)} \right\} \\ 0 \\ 0 \end{bmatrix} \quad . \tag{7-101}$$

Now the rotating part is axisymmetric and so any section of it normal to the axis of rotation must have n-fold symmetry, about the point of intersection of the axis and the section, where n is an integer greater than or equal to 3. Consequently.

$$f_{pf}^{(p)}\left(x_{pf}^{(p)},\ y_{pf}^{(p)} \cos \frac{2m\pi}{n} - z_{pf}^{(p)} \sin \frac{2m\pi}{n},\ z_{pf}^{(p)} \cos \frac{2m\pi}{n} + y_{pf}^{(p)} \sin \frac{2m\pi}{n}\right)$$

$$= f_{pf}^{(p)}\left(x_{pf}^{(p)}, y_{pf}^{(p)}, z_{pf}^{(p)}\right) \cos \frac{2m\pi}{n} - g_{pf}^{(p)}\left(x_{pf}^{(p)}, y_{pf}^{(p)}, z_{pf}^{(p)}\right) \sin \frac{2m\pi}{n} \tag{7-102}$$

and

$$g_{pf}^{(p)}\left(x_{pf}^{(p)},\ y_{pf}^{(p)} \cos \frac{2m\pi}{n} - z_{pf}^{(p)} \sin \frac{2m\pi}{n},\ z_{pf}^{(p)} \cos \frac{2m\pi}{n} + y_{pf}^{(p)} \sin \frac{2m\pi}{n}\right)$$

$$= g_{pf}^{(p)}\left(x_{pf}^{(p)}, y_{pf}^{(p)}, z_{pf}^{(p)}\right) \cos \frac{2m\pi}{n} + f_{pf}^{(p)}\left(x_{pf}^{(p)}, y_{pf}^{(p)}, z_{pf}^{(p)}\right) \sin \frac{2m\pi}{n} \quad . \tag{7-103}$$

Thus, since

$$\sum_{m=1}^{n} \cos \frac{2m\pi}{n} = \sum_{m=1}^{n} \sin \frac{2m\pi}{n} = 0 \qquad (n \geq 2) \tag{7-104}$$

we find the sums of the left hand sides of these expressions are both zero, and so we have

$$\left\{\begin{array}{ll}\sum^{(p)} f_{pf}^{(p)} \times \text{any function of } x_{pf}^{(p)} = 0 & \text{(7-105)}\\ \sum^{(p)} g_{pf}^{(p)} \times \text{any function of } x_{pf}^{(p)} = 0 \quad . & \text{(7-106)}\end{array}\right.$$

These relationships, combined with (7-101), also enable us to deduce that

$$\sum^{(p)} e_{pf}^{(p)} y_{pf}^{(p)} = \sum^{(p)} e_{pf}^{(p)} z_{pf}^{(p)} = 0 \quad . \tag{7-107}$$

This is not all that we can deduce from (7-102) and (7-103). We have

$$\left\{\begin{array}{ll}\sum_{m=1}^{n} \cos^2 \frac{2m\pi}{n} = \sum_{m=1}^{n} \sin^2 \frac{2m\pi}{n} = \frac{n}{2} \qquad (n \geq 3) & \text{(7-108)}\\ \sum_{m=1}^{n} \sin \frac{2m\pi}{n} \cos \frac{2m\pi}{n} = 0 & \text{(7-109)}\end{array}\right.$$

and so we find that

$$\left\{\begin{array}{ll}\sum^{(p)} \left(f_{pf}^{(p)} y_{pf}^{(p)} - g_{pf}^{(p)} z_{pf}^{(p)}\right) \times \text{any function of } x_{pf}^{(p)} = 0 & \text{(7-110)}\\ \sum^{(p)} \left(f_{pf}^{(p)} z_{pf}^{(p)} + g_{pf}^{(p)} y_{pf}^{(p)}\right) \times \text{any function of } x_{pf}^{(p)} = 0 \quad . & \text{(7-111)}\end{array}\right.$$

7.4.1 Generalised propulsive forces

The column vector of the generalised propulsive forces is

$$-\begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix} = \sum^{(p+)} R^T \begin{bmatrix} e_p^{(n)} \\ f_p^{(n)} \\ g_p^{(n)} \end{bmatrix} = R_1^T \sum^{(p)} \begin{bmatrix} e_p^{(n)} \\ f_p^{(n)} \\ g_p^{(n)} \end{bmatrix} + (R_2 - R_1)^T \sum^{(p)} B \begin{bmatrix} e_p^{(n)} \\ f_p^{(n)} \\ g_p^{(n)} \end{bmatrix}$$

$$+ \sum^{(p+)-(p)} R^T \begin{bmatrix} e_p^{(n)} \\ f_p^{(n)} \\ g_p^{(n)} \end{bmatrix} \quad . \tag{7-112}$$

Now, *cf* equations (7-16) and (7-97)

$$\sum^{(p)} B\begin{bmatrix} e_p^{(n)} \\ f_p^{(n)} \\ g_p^{(n)} \end{bmatrix} \approx \sum^{(p)} \left\{ - C R^T_{p_pt} {}^A x^{(p)}_{pf} \begin{bmatrix} e^{(p)}_{pf} \\ f^{(p)}_{pf} \\ g^{(p)}_{pf} \end{bmatrix} + \left(BP^T + \tfrac{1}{2} B \left[\frac{\partial P^T}{\partial x_{pf}} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \quad \frac{\partial P^T}{\partial y_{pf}} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \quad \frac{\partial P^T}{\partial z_{pf}} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right] \right) \times \right.$$

$$\left. \times (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right\} \tag{7-113}$$

In the Appendix we established a set of relationships of the form

$$\frac{\partial \Pi_f}{\partial x_{pf}} \frac{\partial \Pi_f^T}{\partial y_{pf}} = - {}^A a_{pf} {}^A b_{pf} \tag{7-114}$$

and so (*cf* equation (7-95), (7-14) and (7-18))

$$BP^T = \begin{bmatrix} aa & ab & ac \\ ba & bb & bc \\ ca & cb & cc \end{bmatrix} \tag{7-115}$$

where

$$ab = - \begin{bmatrix} x^{(p)}_{pf} & y^{(p)}_{pf} & z^{(p)}_{pf} \end{bmatrix} R_{p_{pt}} {}^A a_{pf} {}^A b_{pf} R^T_{p_{pt}} \begin{bmatrix} e^{(p)}_{pf} \\ f^{(p)}_{pf} \\ g^{(p)}_{pf} \end{bmatrix}$$

$$= - a^T_{pf} {}^A \left(R^T_{p_pt} \begin{bmatrix} x^{(p)}_{pf} \\ y^{(p)}_{pf} \\ z^{(p)}_{pf} \end{bmatrix} \right)^A \left(R^T_{p_{pt}} \begin{bmatrix} e^{(p)}_{pf} \\ f^{(p)}_{pf} \\ g^{(p)}_{pf} \end{bmatrix} \right) b_{pf} \tag{7-116}$$

etc.

Now, using (7-110), (7-111), (7-107), (7-105) and (7-106), we find that

$$\sum^{(p)} \left({}^{A}R^{T}_{p_p} \, t \begin{bmatrix} x^{(p)}_{pf} \\ y^{(p)}_{pf} \\ z^{(p)}_{pf} \end{bmatrix} \right) \left({}^{A}R^{T}_{p_p} \, t \begin{bmatrix} e^{(p)}_{pf} \\ f^{(p)}_{pf} \\ g^{(p)}_{pf} \end{bmatrix} \right)$$

$$= - \sum^{(p)} \begin{bmatrix} y^{(p)}_{pf} f^{(p)}_{pf} + z^{(p)}_{pf} g^{(p)}_{pf} & 0 & 0 \\ 0 & z^{(p)}_{pf} g^{(p)}_{pf} + x^{(p)}_{pf} e^{(p)}_{pf} & - z^{(p)}_{pf} f^{(p)}_{pf} \\ 0 & - y^{(p)}_{pf} g^{(p)}_{pf} & x^{(p)}_{pf} e^{(p)}_{pf} + y^{(p)}_{pf} f^{(p)}_{pf} \end{bmatrix}$$

...... (7-117)

and so*

$$\sum^{(p)} BP^{T} = \frac{\left\{ \sum^{(p)} \left(y^{(p)}_{pf} f^{(p)}_{pf} + z^{(p)}_{pf} g^{(p)}_{pf} \right) \right\}}{(r^2 s^4)_{pf}} \begin{bmatrix} yz \\ -zx \\ 0 \end{bmatrix}_{pf} \begin{bmatrix} yz & -zx & 0 \end{bmatrix}_{pf}$$

$$+ \frac{\left\{ \sum^{(p)} \left(z^{(p)}_{pf} g^{(p)}_{pf} + x^{(p)}_{pf} e^{(p)}_{pf} \right) \right\}}{(r^4 s^2)_{pf}} \begin{bmatrix} -xz \\ -yz \\ s^2 \end{bmatrix}_{pf} \begin{bmatrix} -xz & -yz & s^2 \end{bmatrix}_{pf}$$

$$+ \frac{\left\{ \sum^{(p)} \left(x^{(p)}_{pf} e^{(p)}_{pf} + y^{(p)}_{pf} f^{(p)}_{pf} \right) \right\}}{(r^2 s^2)_{pf}} \begin{bmatrix} y \\ -x \\ 0 \end{bmatrix}_{pf} \begin{bmatrix} y & -x & 0 \end{bmatrix}_{pf} + \frac{\left(\sum^{(p)} z^{(p)}_{pf} f^{(p)}_{pf} \right)}{(r^3)_{pf}} {}^{A}x_{pf}$$

...... (7-118)

---

* $r^2 = x^2 + y^2 + z^2$, $s^2 = x^2 + y^2$, and so, for example

$$(r^2 s^2)_{pf} = \left(x^2_{pf} + y^2_{pf} + z^2_{pf} \right) \left(x^2_{pf} + y^2_{pf} \right) .$$

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Similarly,

$$
\begin{aligned}
\sum^{(p)} g \frac{\partial P^T}{\partial x_{pf}} = {} & \frac{\left\{\sum^{(p)}\left(y_{pf}^{(p)} f_{pf}^{(p)} + z_{pf}^{(p)} g_{pf}^{(p)}\right)\right\}}{(r^4 s^6)_{pf}} \begin{bmatrix} yz \\ -zx \\ 0 \end{bmatrix}_{pf} \begin{bmatrix} -xyz(2r^2+s^2) & z\{x^2(r^2+s^2)-y^2r^2\} & 0 \end{bmatrix}_{pf} \\
& + \frac{\left\{\sum^{(p)}\left(z_{pf}^{(p)} g_{pf}^{(p)} + x_{pf}^{(p)} e_{pf}^{(p)}\right)\right\}}{(r^6 s^4)_{pf}} \begin{bmatrix} -xz \\ -yz \\ s^2 \end{bmatrix}_{pf} \begin{bmatrix} z(2s^2x^2-r^2y^2) & xyz(2s^2+r^2) & xs^2(z^2-s^2) \end{bmatrix}_{pf} \\
& + \frac{\left\{\sum^{(p)}\left(x_{pf}^{(p)} e_{pf}^{(p)} + y_{pf}^{(p)} f_{pf}^{(p)}\right)\right\}}{(r^4 s^4)_{pf}} \begin{bmatrix} y \\ -x \\ 0 \end{bmatrix}_{pf} \begin{bmatrix} -xy(r^2+s^2) & x^2s^2-y^2r^2 & 0 \end{bmatrix}_{pf} \\
& + \frac{\left(\sum^{(p)} z_{pf}^{(p)} f_{pf}^{(p)}\right)}{(r^5 s^4)_{pf}} \left\{ \begin{bmatrix} y \\ -x \\ 0 \end{bmatrix}_{pf} \begin{bmatrix} z(2s^2x^2-r^2y^2) & xyz(2s^2+r^2) & xs^2(z^2-s^2) \end{bmatrix}_{pf} - \begin{bmatrix} -xz \\ -yz \\ s^2 \end{bmatrix}_{pf} \begin{bmatrix} -xy(r^2+s^2) & x^2s^2-y^2r^2 & 0 \end{bmatrix}_{pf} \right\}
\end{aligned}
$$

...... (7-119)

etc. In this last expression, the row matrices are the transposed columns of $\frac{\partial C}{\partial x_{pf}}$ where (*cf* (7-15))

$$
C = \begin{bmatrix} a_{pf}^T \\ b_{pf}^T \\ c_{pf}^T \end{bmatrix} . \qquad (7\text{-}120)
$$

Equations (7-118) and (7-119) could, of course, as a consequence of (7-96), have been written in terms of just three sums:

$$
\sum^{(p)} x_{pf}^{(p)} e_{pf}^{(p)} , \quad \sum^{(p)} y_{pf}^{(p)} f_{pf}^{(p)} , \quad \sum^{(p)} z_{pf}^{(p)} f_{pf}^{(p)}
$$

but all comeliness is then lost. We will not transcribe the other equations similar to (7-119) but merely note that

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$$\frac{\partial C}{\partial x_{pf}} = \begin{bmatrix} -xyz(2r^2+s^2) & z(2x^2s^2-y^2r^2) & -xy(r^2+s^2) \\ z\{x^2(r^2+s^2)-y^2r^2\} & xyz(2s^2+r^2) & x^2s^2-y^2r^2 \\ 0 & xs^2(z^2-s^2) & 0 \end{bmatrix}_{pf}$$

$$\times \operatorname{diag}\left\{\frac{1}{r^3s^4} \quad \frac{1}{r^4s^3} \quad \frac{1}{r^3s^3}\right\}_{pf} \qquad (7\text{-}121)$$

$$\frac{\partial C}{\partial y_{pf}} = \begin{bmatrix} z\{x^2r^2-y^2(r^2+s^2)\} & xyz(2s^2+r^2) & x^2r^2-y^2s^2 \\ xyz(2r^2+s^2) & z(2y^2s^2-x^2r^2) & xy(r^2+s^2) \\ 0 & ys^2(z^2-s^2) & 0 \end{bmatrix}_{pf}$$

$$\times \operatorname{diag}\left\{\frac{1}{r^3s^4} \quad \frac{1}{r^4s^3} \quad \frac{1}{r^3s^3}\right\}_{pf} \qquad (7\text{-}122)$$

and

$$\frac{\partial C}{\partial z_{pf}} = \begin{bmatrix} y & x(z^2-s^2) & -yz \\ -x & y(z^2-s^2) & xz \\ 0 & -2zs^2 & 0 \end{bmatrix}_{pf}$$

$$\times \operatorname{diag}\left\{\frac{1}{r^3} \quad \frac{1}{r^4s} \quad \frac{1}{r^3s}\right\}_{pf} \quad . \qquad (7\text{-}123)$$

To complete the exposition of the second term in the expression (7-112), for the generalised propulsive forces, we require also the following, easily obtained relationship:

$$-\sum^{(p)} C R^{T}_{p_{pt}} A_{x^{(p)}_{pf}} \begin{bmatrix} e^{(p)}_{pf} \\ f^{(p)}_{pf} \\ g^{(p)}_{pf} \end{bmatrix} = 2C \begin{bmatrix} \sum^{(p)} z^{(p)}_{pf} f^{(p)}_{pf} \\ 0 \\ 0 \end{bmatrix} . \qquad (7\text{-}124)$$

For the first term in (7-112) we require the following relationship which is obtained from equation (7-97) using (7-95), (2-4), (7-105), (7-106) and equations (A-5) to (A-14)

$$\sum^{(p)} \begin{bmatrix} e^{(n)}_{p} \\ f^{(n)}_{p} \\ g^{(n)}_{p} \end{bmatrix} = \left(\sum^{(p)} e^{(p)}_{pf}\right) \Bigg[\frac{1}{\sqrt{x^2_{pf} + y^2_{pf} + z^2_{pf}}} \begin{bmatrix} x_{pf} \\ y_{pf} \\ z_{pf} \end{bmatrix}$$

$$+ \Bigg\{ \frac{-1}{\left(x^2_{pf} + y^2_{pf} + z^2_{pf}\right)^{\frac{3}{2}}} A^2_{x_{pf}} + \frac{1}{\left(x^2_{pf} + y^2_{pf} + z^2_{pf}\right)^{\frac{3}{2}}} \times$$

$$\times \left(\frac{1}{2} \begin{bmatrix} 3x(x^2 - r^2) & x(3y^2 - r^2) & x(3z^2 - r^2) \\ y(3x^2 - r^2) & 3y(y^2 - r^2) & y(3z^2 - r^2) \\ z(3x^2 - r^2) & z(3y^2 - r^2) & 3z(z^2 - r^2) \end{bmatrix}_{pf} \left({}^{D}(R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right) \right.$$

$$\left. + \begin{bmatrix} 3xyz & -z(3x^2 - r^2) & y(3z^2 - r^2) \\ z(3y^2 - r^2) & -3xyz & x(3y^2 - r^2) \\ y(3z^2 - r^2) & -x(3z^2 - r^2) & 3xyz \end{bmatrix}_{pf} \left({}^{J}(R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right) \right) \Bigg\} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \Bigg]$$

...... (7-125)

(see equations (3-4) and (4-7) for the definition of the J and D matrices).

The other term in (7-112) involves the summation of the local propulsive forces, acting on the non-rotating parts of the aircraft, multiplied by the modal displacements. If we write* (*cf* (7-92))

$$\sum^{(p+)-(p)} R^T \Pi_f^T R^T_{p_p t} \begin{bmatrix} e^{(p)}_{pf} \\ f^{(p)}_{pf} \\ g^{(p)}_{pf} \end{bmatrix} = -P_f^+ \qquad (7\text{-}126)$$

where, from (7-92), P_f^+ is constant, then we require (see equations (7-97), (7-95) and (7-93)) the following additional relationships which are easily deduced:

$$\sum^{(p+)-(p)} R^T \mathcal{P}^T = -\begin{bmatrix} \dfrac{\partial P_f^+}{\partial x_{pf}} & \dfrac{\partial P_f^+}{\partial y_{pf}} & \dfrac{\partial P_f^+}{\partial z_{pf}} \end{bmatrix} \qquad (7\text{-}127)$$

$$\sum^{(p+)-(p)} R^T \begin{bmatrix} [q_1 \cdots q_n](R_2 - R_1)^T \mathcal{P}_1 \\ [q_1 \cdots q_n](R_2 - R_1)^T \mathcal{P}_2 \\ [q_1 \cdots q_n](R_2 - R_1)^T \mathcal{P}_3 \end{bmatrix} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$= \operatorname{diag}\left\{ [q_1 \cdots q_n](R_2 - R_1)^T \begin{bmatrix} \dfrac{\partial}{\partial x_{pf}} \\ \dfrac{\partial}{\partial y_{pf}} \\ \dfrac{\partial}{\partial z_{pf}} \end{bmatrix} \quad \text{Ditto} \quad \text{Ditto} \quad \cdots \right\} \left(\sum^{(p+)-(p)} R^T \mathcal{P}^T \right) (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$= -\begin{bmatrix} \begin{bmatrix} \dfrac{\partial^2 P_f^+}{\partial x_{pf}^2} & \dfrac{\partial^2 P_f^+}{\partial x_{pf} \partial y_{pf}} & \dfrac{\partial^2 P_f^+}{\partial x_{pf} \partial z_{pf}} \end{bmatrix} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \\ \begin{bmatrix} \dfrac{\partial^2 P_f^+}{\partial x_{pf} \partial y_{pf}} & \dfrac{\partial^2 P_f^+}{\partial y_{pf}^2} & \dfrac{\partial^2 P_f^+}{\partial y_{pf} \partial z_{pf}} \end{bmatrix} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \\ \begin{bmatrix} \dfrac{\partial^2 P_f^+}{\partial x_{pf} \partial z_{pf}} & \dfrac{\partial^2 P_f^+}{\partial y_{pf} \partial z_{pf}} & \dfrac{\partial^2 P_f^+}{\partial z_{pf}^2} \end{bmatrix} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \end{bmatrix} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$= -\left\{ \begin{bmatrix} \dfrac{\partial^2 P_f^+}{\partial x_{pf}^2} & \dfrac{\partial^2 P_f^+}{\partial y_{pf}^2} & \dfrac{\partial^2 P_f^+}{\partial z_{pf}^2} \end{bmatrix}^D \left((R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right) + 2 \begin{bmatrix} \dfrac{\partial^2 P_f^+}{\partial y_{pf} \partial z_{pf}} & \dfrac{-\partial^2 P_f^+}{\partial z_{pf} \partial x_{pf}} & \dfrac{\partial^2 P_f^+}{\partial x_{pf} \partial y_{pf}} \end{bmatrix}^J \left((R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right) \right\} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \qquad (7\text{-}128)$$

---

* The minus sign has been included in this definition for consistency with our definition of the propulsive contributions to the generalised forces as $-P_i \left(= -\left\{ P_{if} + \sum_i P_{ij} q_j + \ldots \right\} \right)$. The elements of P_f^+ will then be seen to be the parts of the P_{if} arising from the propulsive forces on the non-rotating part of the aircraft.

Consequently

$$\sum^{(p+)-(p)} R^T \begin{bmatrix} e_p^{(n)} \\ f_p^{(n)} \\ g_p^{(n)} \end{bmatrix} = -P_f^+ - \left\{ \begin{bmatrix} \dfrac{\partial P_f^+}{\partial x_{pf}} & \dfrac{\partial P_f^+}{\partial y_{pf}} & \dfrac{\partial P_f^+}{\partial z_{pf}} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \dfrac{\partial^2 P_f^+}{\partial x_{pf}^2} & \dfrac{\partial^2 P_f^+}{\partial y_{pf}^2} & \dfrac{\partial^2 P_f^+}{\partial z_{pf}^2} \end{bmatrix} D\left((R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right) \right.$$

$$\left. + \begin{bmatrix} \dfrac{\partial^2 P_f^+}{\partial y_{pf} \partial z_{pf}} & \dfrac{-\partial^2 P_f^+}{\partial z_{pf} \partial x_{pf}} & \dfrac{\partial^2 P_f^+}{\partial x_{pf} \partial y_{pf}} \end{bmatrix} J\left((R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right) \right\} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

...... (7-129)

Incidentally we note, from the Appendix (equations (A-15) to (A-22)) and equation (7-120) that

$$\begin{bmatrix} \dfrac{\partial P_f^+}{\partial x_{pf}} & \dfrac{\partial P_f^+}{\partial y_{pf}} & \dfrac{\partial P_f^+}{\partial z_{pf}} \end{bmatrix} = \sum^{(p+)-(p)} R^T A \left(\Pi_f^T R_{p_p t}^T \begin{bmatrix} e_{pf}^{(p)} \\ f_{pf}^{(p)} \\ g_{pf}^{(p)} \end{bmatrix} \right) C^T$$

$$= \sum^{(p+)-(p)} R^T \Pi_f^T R_{p_p t}^T A_{e_{pf}^{(p)}} R_{p_p t} \Pi_f C^T$$

$$= \sum^{(p+)-(p)} R^T A_{e_{pf}^{(n)}} C^T \qquad (7\text{-}130)$$

where $\left\{ e_{pf}^{(n)} \;\; f_{pf}^{(n)} \;\; g_{pf}^{(n)} \right\}$ are the components, in the unperturbed state, of the local propulsive force referred to the no-deformation-body-fixed axes (*cf* equation (7-92)). Also, from equations (7-10) and (7-16), on the rotating part

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\partial R}{\partial y_{pf}^{(n)}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} C^T (R_2 - R_1) \qquad (7\text{-}131)$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\partial R}{\partial z_{pf}^{(n)}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} C^T(R_2 - R_1) \tag{7-132}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \frac{\partial R}{\partial x_{pf}^{(n)}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} C^T(R_2 - R_1) \tag{7-133}$$

and so, if we write

$$R = \begin{bmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \\ c_1 & \dots & c_n \end{bmatrix} \tag{7-134}$$

then

$$C^T(R_2 - R_1) = \begin{bmatrix} \dfrac{\partial c_1}{\partial y_{pf}^{(n)}} & \dots & \dfrac{\partial c_n}{\partial y_{pf}^{(n)}} \\ \dfrac{\partial a_1}{\partial z_{pf}^{(n)}} & \dots\dots\dots\dots & \\ \dfrac{\partial b_1}{\partial x_{pf}^{(n)}} & \dots\dots\dots\dots & \end{bmatrix} \tag{7-135}$$

which shows that $C^T(R_2 - R_1)$ is akin to the modal slope matrix P_q of earlier work (Ref 2, equation (11)) but for a point on the rotating part rather than the origin of the body-fixed axes.

Putting all the results together, we therefore find that

$$
-\begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix} = \frac{\left(\sum^{(p)} e_{p\ell}^{(p)}\right) R_1^T}{\sqrt{x_{p\ell}^2 + y_{p\ell}^2 + z_{p\ell}^2}} \begin{bmatrix} x_{p\ell} \\ y_{p\ell} \\ z_{p\ell} \end{bmatrix} + \frac{2\left(\sum^{(p)} z_{p\ell}^{(p)} f_{p\ell}^{(p)}\right)(R_2 - R_1)^T}{\left(x_{p\ell}^2 + y_{p\ell}^2\right)\sqrt{x_{p\ell}^2 + y_{p\ell}^2 + z_{p\ell}^2}} \begin{bmatrix} y_{p\ell} z_{p\ell} \\ -z_{p\ell} x_{p\ell} \\ 0 \end{bmatrix} - P_\ell^+
$$

$$
+ \left\{ \frac{-\left(\sum^{(p)} e_{p\ell}^{(p)}\right)}{\left(x_{p\ell}^2 + y_{p\ell}^2 + z_{p\ell}^2\right)^{2}} R_1^T A_{x_{p\ell}}^2 + (R_2 - R_1)^T \times \right.
$$

$$
\times \left(\frac{2\left(\sum^{(p)} y_{p\ell}^{(p)} f_{p\ell}^{(p)}\right)}{(r^2 s^4)_{p\ell}} \begin{bmatrix} y^2 z^2 & -xyz^2 & 0 \\ -xyz^2 & z^2 x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{p\ell} \right.
$$

$$
\left. + \frac{\left(\sum^{(p)} \left(x_{p\ell}^{(p)} e_{p\ell}^{(p)} + y_{p\ell}^{(p)} f_{p\ell}^{(p)}\right)\right)}{(r^4)_{p\ell}} \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix}_{p\ell} + \frac{\left(\sum^{(p)} z_{p\ell}^{(p)} f_{p\ell}^{(p)}\right)}{(r^3)_{p\ell}} A_{x_{p\ell}} \right)
$$

$$
\left. - \begin{bmatrix} \frac{\partial P_\ell^+}{\partial x_{p\ell}} & \frac{\partial P_\ell^+}{\partial y_{p\ell}} & \frac{\partial P_\ell^+}{\partial z_{p\ell}} \end{bmatrix} \right\} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}
$$

$$
+ \left\{ \frac{\left(\sum^{(p)} e_{p\ell}^{(p)}\right)}{2(r^5)_{p\ell}} R_1^T \begin{bmatrix} 3x(x^2 - r^2) & x(3y^2 - r^2) & x(3z^2 - r^2) \\ y(3x^2 - r^2) & 3y(y^2 - r^2) & y(3z^2 - r^2) \\ z(3x^2 - r^2) & z(3y^2 - r^2) & 3z(z^2 - r^2) \end{bmatrix}_{p\ell} + \frac{(R_2 - R_1)^T}{2} \right.
$$

$$
\times \left(\frac{2\left(\sum^{(p)} y_{p\ell}^{(p)} f_{p\ell}^{(p)}\right)}{(r^4 s^6)_{p\ell}} \begin{bmatrix} -xy^2 z^2 (2r^2 + s^2) & xy^2 z^2 (2r^2 + s^2) & 0 \\ x^2 yz^2 (2r^2 + s^2) & -x^2 yz^2 (2r^2 + s^2) & 0 \\ 0 & 0 & 0 \end{bmatrix}_{p\ell} \right.
$$

$$
+ \frac{\left(\sum^{(p)} \left(x_{p\ell}^{(p)} e_{p\ell}^{(p)} + y_{p\ell}^{(p)} f_{p\ell}^{(p)}\right)\right)}{(r^6 s^2)_{p\ell}} \begin{bmatrix} -2xs^2(y^2 + z^2) & x(r^2 s^2 + 2y^2 s^2) & 2xs^2 s^2 \\ y(r^2 s^2 + 2x^2 s^2) & -2ys^2(x^2 + z^2) & 2ys^2 s^2 \\ z(2s^2 x^2 - r^2 y^2) & z(2s^2 y^2 - r^2 x^2) & -2zs^4 \end{bmatrix}_{p\ell}
$$

$$
+ \frac{\left(\sum^{(p)} z_{pf}^{(p)} f_{pf}^{(p)}\right)}{(r^5 s^2)_{pf}} \begin{bmatrix} -yz(y^2+z^2) & yz(2y^2+x^2) & -2yzs^2 \\ -xz(2x^2+y^2) & xz(x^2+z^2) & 2xzs^2 \\ xz(r^2+s^2) & -xy(r^2+s^2) & 0 \end{bmatrix}_{pf} \Bigg)
$$

$$
- \frac{1}{2} \left[\frac{\partial^2 P_f^+}{\partial x_{pf}^2} \quad \frac{\partial^2 P_f^+}{\partial y_{pf}^2} \quad \frac{\partial^2 P_f^+}{\partial z_{pf}^2} \right] \Bigg\}^{D} \left((R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right)^{(R_2 - R_1)} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}
$$

$$
+ \left\{ \frac{\left(\sum^{(p)} e_{pf}^{(p)}\right)}{(r^5)_{pf}} R_1^T \begin{bmatrix} 3xyz & -z(3x^2-r^2) & y(3x^2-r^2) \\ z(3y^2-r^2) & -3xyz & x(3y^2-r^2) \\ y(3z^2-r^2) & -x(3z^2-r^2) & 3xyz \end{bmatrix}_{pf} + \frac{(R_2 - R_1)^T}{2} \times \right.
$$

$$
\times \left(\frac{2\left(\sum^{(p)} y_{pf}^{(p)} f_{pf}^{(p)}\right)}{(r^4 s^6)_{pf}} \begin{bmatrix} -xyzs^4 & -zy^2 s^4 & yz^2(x^2-y^2)(2r^2+s^2) \\ zx^2 s^4 & xyzs^4 & -xz^2(x^2-y^2)(2r^2+s^2) \\ 0 & 0 & 0 \end{bmatrix}_{pf} \right.
$$

$$
+ \frac{\left(\sum^{(p)} \left(x_{pf}^{(p)} e_{pf}^{(p)} + y_{pf}^{(p)} f_{pf}^{(p)}\right)\right)}{(r^6 s^2)_{pf}} \begin{bmatrix} xyz(3s^2-z^2) & z\{y^2 r^2 + 2x^2(z^2-s^2)\} & y(4x^2 s^2 - r^2 s^2 - r^4) \\ z\{x^2 r^2 - 2y^2(z^2-s^2)\} & xyz(r^2-4s^2) & x(4y^2 s^2 - r^2 s^2 - r^4) \\ 2ys^2(z^2-s^2) & -2xs^2(z^2-s^2) & 2xyz(2s^2+r^2) \end{bmatrix}_{pf}
$$

$$
+ \frac{\left(\sum^{(p)} z_{pf}^{(p)} f_{pf}^{(p)}\right)}{(r^5 s^2)_{pf}} \begin{bmatrix} 2y^2(z^2-s^2) + x^2 z^2 & xy(2s^2-z^2) & xz(2x^2+4y^2+z^2) \\ xy(2s^2-z^2) & 2x^2(z^2-s^2) + y^2 z^2 & -yz(4x^2+2y^2+z^2) \\ -xzs^2 & -yzs^2 & -(x^2-y^2)(r^2+s^2) \end{bmatrix}_{pf} \Bigg)
$$

$$
- \left[\frac{\partial^2 P_f^+}{\partial y_{pf} \partial z_{pf}} \quad \frac{-\partial^2 P_f^+}{\partial z_{pf} \partial x_{pf}} \quad \frac{\partial^2 P_f^+}{\partial x_{pf} \partial y_{pf}} \right] \Bigg\}^{J} \left((R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right)^{(R_2 - R_1)} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \qquad (7\text{-}136)
$$

where P_f^+ is given by equation (7-127), and as a consequence of (7-92) is not time dependent.

7.4.2 Overall propulsive forces

We have

$$\begin{bmatrix} X_p^{(n)} \\ Y_p^{(n)} \\ Z_p^{(n)} \end{bmatrix} = \sum^{(p+)} \begin{bmatrix} e_p^{(n)} \\ f_p^{(n)} \\ g_p^{(n)} \end{bmatrix} \tag{7-137}$$

and

$$\begin{bmatrix} L_{pn}^{(n)} \\ M_{pn}^{(n)} \\ N_{pn}^{(n)} \end{bmatrix} = \sum^{(p+)} A_{x_n^{(n)}} \begin{bmatrix} e_p^{(n)} \\ f_p^{(n)} \\ g_p^{(n)} \end{bmatrix} \quad . \tag{7-138}$$

These two expressions bear a *prima facie* resemblance to the expressions for the generalised propulsive forces (equation (7-112)). However, while the unit matrix can be considered as a particular case of R – it satisfies the condition (7-4) for then $R_2 = R_1$ – , $A_{x_n^{(n)}}$ cannot since the $x_n^{(n)}$ etc are functions of the generalised coordinates q_i (equation (7-17)). We can therefore write down immediately, by comparison with (7-136), the overall propulsive forces as

$$\begin{bmatrix} X_p^{(n)} \\ Y_p^{(n)} \\ Z_p^{(n)} \end{bmatrix} = \begin{bmatrix} X_{pf}^+ \\ Y_{pf}^+ \\ Z_{pf}^+ \end{bmatrix} + \frac{\left(\sum^{(p)} e_{pf}^{(p)}\right)}{\sqrt{x_{pf}^2 + y_{pf}^2 + z_{pf}^2}} \begin{bmatrix} x_{pf} \\ y_{pf} \\ z_{pf} \end{bmatrix} \tag{7-139}$$

where

$$\begin{bmatrix} X_{pf}^+ \\ Y_{pf}^+ \\ Z_{pf}^+ \end{bmatrix} = \Pi_f^T \sum^{(p+)-(p)} R_{p_p t}^T \begin{bmatrix} e_{pf}^{(p)} \\ f_{pf}^{(p)} \\ g_{pf}^{(p)} \end{bmatrix} = \sum^{(p+)-(p)} \begin{bmatrix} e_{pf}^{(n)} \\ f_{pf}^{(n)} \\ g_{pf}^{(n)} \end{bmatrix} \tag{7-140}$$

and is a constant (*cf* (7-92)).

The moments about the origin of the no-deformation-body-fixed axes require further analysis. Now, on the rotating part $\left\{x_n^{(n)} \ y_n^{(n)} \ z_n^{(n)}\right\}$ is given by equations (7-17) and (7-18) while for $\left\{e_p^{(n)} \ f_p^{(n)} \ g_n^{(n)}\right\}$ we turn to equation (7-97). By comparison with (7-16) it is clear that

$$R_{p_p t} \Pi_f \mathcal{P}^T = A_{e_{pf}^{(p)}} R_{p_p t} \mathcal{C}^T \quad . \tag{7-141}$$

Thus we immediately have the following relationship which we require

$$A_{x_{pf}^{(p)}} R_{p_p t} \Pi_f \begin{bmatrix} \frac{\partial \mathcal{P}^T}{\partial x_{pf}} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} & \frac{\partial \mathcal{P}^T}{\partial y_{pf}} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} & \frac{\partial \mathcal{P}^T}{\partial z_{pf}} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \end{bmatrix}$$

$$= A_{x_{pf}^{(p)}} A_{e_{pf}^{(p)}} R_{p_p t} \begin{bmatrix} \frac{\partial \mathcal{C}^T}{\partial x_{pf}} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} & \frac{\partial \mathcal{C}^T}{\partial y_{pf}} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} & \frac{\partial \mathcal{C}^T}{\partial z_{pf}} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \end{bmatrix} .$$

...... (7-142)

In addition, using (7-88) and (7-99)

$$\sum\nolimits^{(p)} \mathcal{P}^T = \frac{-\sum^{(p)} e_{pf}^{(p)}}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{3}{2}}} A_{x_{pf}}^2 \tag{7-143}$$

while, from (7-105) to (7-107), (7-110) and (7-111),

$$\sum\nolimits^{(p)} A_{x_{pf}^{(p)}} \begin{bmatrix} e_{pf}^{(p)} \\ f_{pf}^{(p)} \\ g_{pf}^{(p)} \end{bmatrix} = -2\left(\sum\nolimits^{(p)} f_{pf}^{(p)} z_{pf}^{(p)}\right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{7-144}$$

and

$$\sum^{(p)} A_{x^{(p)}_{pf}} A_{e^{(p)}_{pf}} = -\left(\sum^{(p)} e^{(p)}_{pf} x^{(p)}_{pf}\right) \mathrm{diag}\{0 \quad 1 \quad 1\}$$

$$- \left(\sum^{(p)} f^{(p)}_{pf} y^{(p)}_{pf}\right) \mathrm{diag}\{2 \quad 1 \quad 1\}$$

$$- \left(\sum^{(p)} f^{(p)}_{pf} z^{(p)}_{pf}\right) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} . \qquad (7\text{-}145)$$

With a little manipulation one can then show that

$$\sum^{(p)} A_{\left(B^T(R_2-R_1)\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}\right)} P^T(R_2-R_1)\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$= \Pi_f^T \left\{ R^T_{p_{pt}} \left(\sum^{(p)} A_{x^{(p)}_{pf}} \begin{bmatrix} e^{(p)}_{pf} \\ f^{(p)}_{pf} \\ g^{(p)}_{pf} \end{bmatrix} \right) [q_1 \quad \cdots \quad q_n](R_2 - R_1)^T C \right.$$

$$\left. + A_{\left(C^T(R_2-R_1)\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}\right)} R^T_{p_p t} \sum^{(p)} \left(\begin{bmatrix} x^{(p)}_{pf} \\ y^{(p)}_{pf} \\ z^{(p)}_{pf} \end{bmatrix} \begin{bmatrix} e^{(p)}_{pf} & f^{(p)}_{pf} & g^{(p)}_{pf} \end{bmatrix} - A_{x^{(p)}_{pf}} A_{e^{(p)}_{pf}} \right) R_{p_p t} \right\} C^T(R_2 - R_1)\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$= \frac{-2\left(\sum^{(p)} f^{(p)}_{pf} z^{(p)}_{pf}\right)}{\left(x^2_{pf} + y^2_{pf} + z^2_{pf}\right)^{\frac{1}{2}}} \begin{bmatrix} x_{pf} \\ y_{pf} \\ z_{pf} \end{bmatrix} [q_1 \quad \cdots \quad q_n](R_2 - R_1)^T C C^T (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \qquad (7\text{-}146)$$

for

$$\sum^{(p)} \left(\begin{bmatrix} x^{(p)}_{pf} \\ y^{(p)}_{pf} \\ z^{(p)}_{pf} \end{bmatrix} \begin{bmatrix} e^{(p)}_{pf} & f^{(p)}_{pf} & g^{(p)}_{pf} \end{bmatrix} - A_{x^{(p)}_{pf}} A_{e^{(p)}_{pf}} \right) = \sum^{(p)} \left(e^{(p)}_{pf} x^{(p)}_{pf} + 2 f^{(p)}_{pf} y^{(p)}_{pf} \right) I \ .$$

$$\ldots\ldots \ (7\text{-}147)$$

Thus we ultimately find that, using (7-99), (7-7), (7-88)

$$
\begin{aligned}
\sum^{(p)} A_{x_n^{(n)}} \begin{bmatrix} e_p^{(n)} \\ f_p^{(n)} \\ g_p^{(n)} \end{bmatrix} = & \left\{ \frac{\left(\sum^{(p)} e_{pf}^{(p)}\right)}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{1}{2}}} A_{x_{f1}} - \frac{2\left(\sum^{(p)} f_{pf}^{(p)} z_{pf}^{(p)}\right)}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{1}{2}}} I \right\} \begin{bmatrix} x_{pf} \\ y_{pf} \\ z_{pf} \end{bmatrix} - \left\{ \frac{\left(\sum^{(p)} e_{pf}^{(p)}\right)}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{1}{2}}} \left(A_{x_{pf}} R_1 + \frac{A_{x_{f1}} A^2_{x_{pf}}}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)} (R_2 - R_1) \right) \right. \\
& \left. - \frac{2\left(\sum^{(p)} f_{pf}^{(p)} z_{pf}^{(p)}\right)}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{3}{2}}} A^2_{x_{pf}} (R_2 - R_1) \right\} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \\
& + \left(\frac{1}{2} \left\{ \frac{\sum^{(p)} e_{pf}^{(p)}}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{1}{2}}} A_{x_{f1}} A_{x_{pf}} \Pi_f^T - \left(\sum^{(p)} e_{pf}^{(p)} x_{pf}^{(p)}\right) \Pi_f^T \operatorname{diag}\{1 \quad 0 \quad 0\} - \left(\sum^{(p)} f_{pf}^{(p)} y_{pf}^{(p)}\right) \Pi_f^T \operatorname{diag}\{2 \quad 1 \quad 1\} \right. \right. \\
& \left. - \left(\sum^{(p)} f_{pf}^{(p)} z_{pf}^{(p)}\right) \Pi_f^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right\} \left[\frac{\partial C^T}{\partial x_{pf}} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \quad \frac{\partial C^T}{\partial y_{pf}} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \quad \frac{\partial C^T}{\partial z_{pf}} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right] \\
& - \frac{\left(\sum^{(p)} e_{pf}^{(p)}\right)}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{3}{2}}} A \left(R_1 \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right) A^2_{x_{pf}} \\
& \left. - \frac{2\left(\sum^{(p)} f_{pf}^{(p)} z_{pf}^{(p)}\right)}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{1}{2}}} \begin{bmatrix} x_{pf} \\ y_{pf} \\ z_{pf} \end{bmatrix} [q_1 \; \cdots \; q_n] (R_2 - R_1)^T C C^T \right) (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \quad . \qquad (7\text{-}148)
\end{aligned}
$$

Writing

$$
\begin{bmatrix} L^+_{pnf} \\ M^+_{pnf} \\ N^+_{pnf} \end{bmatrix} = \sum^{(p+)-(p)} A_{x_f} \Pi_f^T R^T_{p_p t} \begin{bmatrix} e^{(p)}_{pf} \\ f^{(p)}_{pf} \\ g^{(p)}_{pf} \end{bmatrix} \qquad (7\text{-}149)
$$

$$
R = - \sum^{(p+)-(p)} \Pi_f^T R^T_{p_p t} A_{e^{(p)}_{pf}} R_{p_p t} \Pi_f R \qquad (7\text{-}150)
$$

it is easily shown, in a similar manner to that used for the contributions to the generalised forces from the propulsive forces on the non-rotating parts, that the same forces produce moments:

$$\begin{aligned}\sum^{(p+)-(p)} {}^{A}x_n^{(n)}\begin{bmatrix} e_p^{(n)} \\ f_p^{(n)} \\ g_p^{(n)} \end{bmatrix} \approx \begin{bmatrix} L^+_{pnf} \\ M^+_{pnf} \\ N^+_{pnf} \end{bmatrix} + \Biggl\{ R + \begin{bmatrix} \dfrac{\partial}{\partial x_{pf}} \begin{bmatrix} L^+_{pnf} \\ M^+_{pnf} \\ N^+_{pnf} \end{bmatrix} & \dfrac{\partial}{\partial y_{pf}} \begin{bmatrix} L^+_{pnf} \\ M^+_{pnf} \\ N^+_{pnf} \end{bmatrix} & \dfrac{\partial}{\partial z_{pf}} \begin{bmatrix} L^+_{pnf} \\ M^+_{pnf} \\ N^+_{pnf} \end{bmatrix} \end{bmatrix} (R_2 - R_1) \Biggr\} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \\
+ \Biggl\{ \begin{bmatrix} \dfrac{\partial R}{\partial x_{pf}} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} & \dfrac{\partial R}{\partial y_{pf}} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} & \dfrac{\partial R}{\partial z_{pf}} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \end{bmatrix} \\
+ \frac{1}{2} \begin{bmatrix} \dfrac{\partial^2}{\partial x_{pf}^2} \begin{bmatrix} L^+_{pnf} \\ M^+_{pnf} \\ N^+_{pnf} \end{bmatrix} & \dfrac{\partial^2}{\partial y_{pf}^2} \begin{bmatrix} L^+_{pnf} \\ M^+_{pnf} \\ N^+_{pnf} \end{bmatrix} & \dfrac{\partial^2}{\partial z_{pf}^2} \begin{bmatrix} L^+_{pnf} \\ M^+_{pnf} \\ N^+_{pnf} \end{bmatrix} \end{bmatrix} {}^{D}\!\left((R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right) \\
+ \begin{bmatrix} \dfrac{\partial^2}{\partial y_{pf}\partial z_{pf}} \begin{bmatrix} L^+_{pnf} \\ M^+_{pnf} \\ N^+_{pnf} \end{bmatrix} & \dfrac{-\partial^2}{\partial z_{pf}\partial x_{pf}} \begin{bmatrix} L^+_{pnf} \\ M^+_{pnf} \\ N^+_{pnf} \end{bmatrix} & \dfrac{\partial^2}{\partial x_{pf}\partial y_{pf}} \begin{bmatrix} L^+_{pnf} \\ M^+_{pnf} \\ N^+_{pnf} \end{bmatrix} \end{bmatrix} {}^{J}\!\left((R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right) \Biggr\} (R_2 - R_1) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \quad .\end{aligned}$$

...... (7-151)

The expressions for the total moments are therefore given by adding this equation to (7-148). Because of their length we will not give them here.

8 STRUCTURAL FORCES

Resulting from such things as the stresses in the material, friction between different components of the structure, etc, there will be a force on any particle of the aircraft which can be called a structural force. Calling the components of this structural force vector $\left(e_s^{(n)}, f_s^{(n)}, g_s^{(n)}\right)$, referred to the no-deformation-body-fixed axes, we write them in the form:

$$\begin{bmatrix} e_s^{(n)} \\ f_s^{(n)} \\ g_s^{(n)} \end{bmatrix} = \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} + \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \quad . \qquad (8\text{-}1)$$

This assumes a linear relationship between stress and strain, is consequent upon the linear relationship between a particle's cartesian coordinates and the generalised coordinates (equation (2-1)), and approximates to any contribution not resulting from a stress by a contribution of the above form. The coefficients, e_{si} etc, may be differential operators.

The structure cannot exert any overall force or moment on itself, and so it is easily shown (*cf* Appendix A of Ref 2) that the following conditions must be satisfied:

$$\sum \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} = 0 \tag{8-2}$$

$$\sum A_{x_f} \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} = 0 \tag{8-3}$$

$$\sum \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots\cdot & \\ g_{s1} & \cdots\cdots\cdot & \end{bmatrix} = 0 \tag{8-4}$$

$$\sum A_{x_f} \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots\cdot & \\ g_{s1} & \cdots\cdots\cdot & \end{bmatrix} = \sum A_{e_{sf}} R \tag{8-5}$$

$$\sum A_{e_{si}} R = 0 \qquad \text{for all } i\,. \tag{8-6}$$

The last of these equations, (8-6), comes from equating to zero the second order terms, in the generalised coordinates, in the expression for the moment about, say, the origin of the no-deformation-body-fixed axes. If we had included some second order terms in the expression for the structural force (equation (8-1)) this condition (8-6) would no longer hold though the other four would.

8.1 Generalised structural forces

The column vector of the generalised structural forces is

$$-\begin{bmatrix}E_1\\ \vdots \\ E_n\end{bmatrix} = \sum R^T \begin{bmatrix} e_s^{(n)} \\ f_s^{(n)} \\ g_s^{(n)}\end{bmatrix}$$

$$= \sum R^T \begin{bmatrix} e_{sf}\\ f_{sf}\\ g_{sf}\end{bmatrix} + R^T \begin{bmatrix} e_{s1} & \cdots & e_{sn}\\ f_{s1} & \cdots\cdots\cdots & \\ g_{s1} & \cdots\cdots\cdots & \end{bmatrix}\begin{bmatrix}q_1\\ \vdots\\ q_n\end{bmatrix} . \qquad (8\text{-}7)$$

8.2 Overall structural forces

The overall translational forces and moments will, as already stated, be zero whatever the perturbation and datum state. This fact has indeed already been used to derive the conditions (equations (8-2) to (8-6)) which are satisfied by the coefficients in the expression (8-1), for the structural force vector.

9 EQUATIONS OF EQUILIBRIUM

The equations of motion will of course be satisfied whatever the perturbation from the datum state and in particular when the perturbations are zero. The equations then obtained, we describe as the equations of equilibrium or datum motion equations. In this state the effective forces are all zero (*cf* equations (4-16) to (4-18), (7-71), (7-86) and (7-90)), and so the equations merely state that the sums of all the contributions to the generalised and overall forces are all zero. Thus we have

$$\left[- (Q_i)_f + (G_i)_f + (P_i)_f + (E_i)_f\right] = 0 \qquad (9\text{-}1)$$

where, from (5-11), (5-12) and (5-13)

$$\left[(Q_i)_f\right] = \begin{bmatrix} \sum R^T \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} \\ \sum \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} \\ \sum A_{x_f} \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} \end{bmatrix} \tag{9-2}$$

from (6-8), (6-9) and (6-10)

$$-\left[(G_i)_f\right] = g \begin{bmatrix} \left(\sum \delta m R^T\right) \ell_{\Phi_f} \\ m\ell_{\Phi_f} \\ \left(\sum \delta m A_{x_f}\right) \ell_{\Phi_f} \end{bmatrix} \tag{9-3}$$

from (7-136), (7-139), (7-148) and (7-151)

$$-\left[(P_i)_f\right] = \begin{bmatrix} -P_f^+ \\ \begin{bmatrix} X_{pf}^+ \\ Y_{pf}^+ \\ Z_{pf}^+ \end{bmatrix} \\ \begin{bmatrix} L_{pnf}^+ \\ M_{pnf}^+ \\ N_{pnf}^+ \end{bmatrix} \end{bmatrix} + \frac{\left(\sum^{(p)} e_{pf}^{(p)}\right)}{\sqrt{x_{pf}^2 + y_{pf}^2 + z_{pf}^2}} \begin{bmatrix} R_1^T \begin{bmatrix} x_{pf} \\ y_{pf} \\ z_{pf} \end{bmatrix} \\ \begin{bmatrix} x_{pf} \\ y_{pf} \\ z_{pf} \end{bmatrix} \\ A_{x_{f1}} \begin{bmatrix} x_{pf} \\ y_{pf} \\ z_{pf} \end{bmatrix} \end{bmatrix} + \frac{2\left(\sum^{(p)} z_{pf}^{(p)} f_{pf}^{(p)}\right)}{\sqrt{x_{pf}^2 + y_{pf}^2 + z_{pf}^2}} \begin{bmatrix} \frac{(R_2 - R_1)^T}{x_{pf}^2 + y_{pf}^2} \begin{bmatrix} y_{pf} z_{pf} \\ -z_{pf} x_{pf} \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ -\begin{bmatrix} x_{pf} \\ y_{pf} \\ z_{pf} \end{bmatrix} \end{bmatrix}$$

......(9-4)

and from (8-7)

$$-\left[(E_i)_f\right] = \begin{bmatrix} \sum R^T \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} \\ 0 \\ 0 \end{bmatrix} . \qquad (9-5)$$

The contribution to the propulsive forces from the non-rotating part, denoted above (equation (9-4)) by all-embracing symbols, is, from (7-126), (7-140) and (7-149)

$$\begin{bmatrix} -P_f^+ \\ \begin{bmatrix} X_{pf}^+ \\ Y_{pf}^+ \\ Z_{pf}^+ \end{bmatrix} \\ \begin{bmatrix} L_{pnf}^+ \\ M_{pnf}^+ \\ N_{pnf}^+ \end{bmatrix} \end{bmatrix} = \sum^{(p+)-(p)} \begin{bmatrix} R^T \Pi_f^T R_{p_{pt}}^T \begin{bmatrix} e_{pf}^{(p)} \\ f_{pf}^{(p)} \\ g_{pf}^{(p)} \end{bmatrix} \\ \Pi_f^T R_{p_{pt}}^T \begin{bmatrix} e_{pf}^{(p)} \\ f_{pf}^{(p)} \\ g_{pf}^{(p)} \end{bmatrix} \\ A_{x_f} \Pi_f^T R_{p_{pt}}^T \begin{bmatrix} e_{pf}^{(p)} \\ f_{pf}^{(p)} \\ g_{pf}^{(p)} \end{bmatrix} \end{bmatrix} = \sum^{(p+)-(p)} \begin{bmatrix} R^T \begin{bmatrix} e_{pf}^{(n)} \\ f_{pf}^{(n)} \\ g_{pf}^{(n)} \end{bmatrix} \\ \begin{bmatrix} e_{pf}^{(n)} \\ f_{pf}^{(n)} \\ g_{pf}^{(n)} \end{bmatrix} \\ A_{x_f} \begin{bmatrix} e_{pf}^{(n)} \\ f_{pf}^{(n)} \\ g_{pf}^{(n)} \end{bmatrix} \end{bmatrix}$$

...... (9-6)

where the local propulsive forces satisfy (7-92).

9.1 Other equilibrium states

Once non-linear terms are introduced into the equations of motion there is always the possibility of equilibrium states, other than the datum state when all the perturbations are zero. Any such equilibrium states will be given by solutions of the equations of motion which satisfy the condition that all the generalised coordinates are constant. Thus, for example, if we had no deformational freedoms (a rigid aircraft), possible equilibrium states are given approximately by the solutions of the equation (*cf* equations (5-14), (5-15), (6-9) and (6-10)):

$$\left[\begin{array}{l} -\begin{bmatrix} X_{\dot{x}} & X_{\dot{y}} & X_{\dot{z}} \\ Y_{\dot{x}} & \cdots\cdots & \\ Z_{\dot{x}} & \cdots\cdots & \end{bmatrix}_{qs} \left(A_{u_f} + J^T_{u_f} J_\phi - \tfrac{1}{2} C^T_{u_f} D_\phi\right) - mg\left(A_{\ell_{\phi_f}} + J^T_{\ell_{\phi_f}} J_\phi - \tfrac{1}{2} C^T_{\ell_{\phi_f}} D_\phi\right) \\ \\ -\begin{bmatrix} L_{\dot{x}} & L_{\dot{y}} & L_{\dot{z}} \\ M_{\dot{x}} & \cdots\cdots & \\ N_{\dot{x}} & \cdots\cdots & \end{bmatrix}_{qs} \left(A_{u_f} + J^T_{u_f} J_\phi - \tfrac{1}{2} C^T_{u_f} D_\phi\right) \end{array}\right] \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = 0 \qquad \text{...... (9-7)}$$

where the subscript qs denotes the quasi-steady values. The above equation neglects any third or higher order terms in the generalised coordinates. No doubt this often will not be good enough for the determination of any equilibrium state which is not the datum state. In such circumstances a better approximation to the local aerodynamic force vector than (5-8) will pretty certainly be required; and so an entirely different approach, such as solving the aerodynamic-gravitational-propulsive-structural-dynamic problem as one problem rather than doing aerodynamic calculations etc first to provide data for the dynamic problem.

It should also be noted that if an autonomous system such as ours has two equilibrium states, one of which is stable then the other one will be unstable. However one can have two equilibrium states both of which are unstable*.

---

* At an equilibrium point the eigenvalue equation has the form $\lambda^{2n} + a_1\lambda^{2n-1} + \ldots + a_{2n} = 0$. If there are just two such points then a_{2n} at one will have the opposite sign to a_{2n} at the other. Therefore all the eigenvalues at both points cannot all have negative or zero real parts.

10 PERTURBATION MOTION EQUATIONS

The equations of motion for perturbations from the datum state clearly follow from (4-1), (4-17) and (4-18) when all the contributions, to the various constituents of the generalised and overall forces, which are independent of the generalised coordinates are omitted (*cf* (9-1)). In the previous sections we have obtained expressions for the various elements in these equations. The relevant contributions to the effective forces are displayed in equations (4-16) to (4-18), (7-71), (7-86) and (7-90); while those to the applied forces (generalised and overall) are found in equations (5-11), (5-14), (5-15), (6-8) to (6-10), (7-136), (7-139), (7-148), (7-151) and (8-7). There is of course no structural contribution to the overall forces. Little would be achieved, other than indigestion, if we were to gather together here all these various contributions, except in the case of small perturbations which we will consider in a moment.

However it is perhaps instructive at this stage to consider briefly the sort of behaviour that might be predicted by our non-linear equations. For example take a one degree of freedom system whose equation is

$$\ddot{q} + b\dot{q} + cq + dq^2 = 0 \qquad b, c > 0. \quad (10\text{-}1)$$

It has two equilibrium states:

(i) the origin $q = 0$ which is a point of asymptotic stability; and

(ii) the point $q = -c/d$ which is a saddle point and therefore unstable.

The trajectories, in the $(q,\dot{q})$ plane then take the form shown in Fig 1 (see Ref 13, p.62). It will be seen that the region of asymptotic stability is bounded.

A second example is the one degree of freedom system whose equation is

$$\ddot{q} + h\dot{q}^2 + cq = 0 \qquad c > 0. \quad (10\text{-}2)$$

In this case we have only one equilibrium state ($q = 0$) but the region of stability is still bounded (*cf* Ref 14, p.104) - see Fig 2. The same sort of thing is true for the equation

$$\ddot{q} + kq\dot{q} + cq = 0 \qquad c > 0. \quad (10\text{-}3)$$

(see Fig 3). In both these cases ((10-2) and (10-3)) the datum (equilibrium) state is stable but not asymptotically stable, but it is possible to have a bounded region of stability even when there is only one equilibrium state and that is asymptotically stable. This is illustrated by the example

$$\ddot{q} + (b\dot{q} + cq)\left(1 + \frac{k}{c}\dot{q}\right) = 0 \qquad k,b,c > 0, \quad (10\text{-}4)$$

for which the region of asymptotic stability is bounded by $\dot{q} = -c/k$ (see Fig 4).

Another possibility, often encountered in physical systems, is the occurrence of limit-cycles. Linear theory then predicts the equilibrium state to be unstable but non-linear theory shows that it is stable in that any path starting in the vicinity of the equilibrium point tends ultimately to a closed curve surrounding that point. This path is then a stable limit-cycle*. Such solutions, in the one degree of freedom case, will not be predicted by our approximation. It is necessary to include third order terms at least. With more than one degree of freedom one cannot say that no limit cycles are possible with present approximation but here again one would intuitively think that a better approximation is required.

To sum up therefore, one may say that the non-linear equations, as developed here, may be useful to indicate whether there is a boundary to the region of stability, but probably one of little value in determining whether an unstable equilibrium state is 'really' unstable or possesses a stable limit-cycle.

10.1 The linear approximation

When we collect together from sections 4.1, 4.2, 5.1, 5.2, 6.1, 6.2, 7.2, 7.3, 7.4.1 and 7.4.2 the various contributions to the applied and effective generalised and overall forces we find that for small perturbations the following equation must be satisfied:

$$\left\{[A_{ij}]D^2 + [J_{ij}]D + [V_{ij}] + [G_{ij}] + [P_{ij}] + [E_{ij}] - [Q_{ij}]\right\}\begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} = 0 \qquad (10\text{-}5)$$

* There is also the possibility of unstable limit-cycles. The equilibrium point is then asymptotically stable - any path starting within the limit-cycle will finish up at the equilibrium point. Semi-stable limit-cycles, stable on one side, unstable on the other, should not occur in practice.

where the various constituent matrices are:

(i) Ponderous inertia

$$[A_{ij}] = \begin{bmatrix} \sum \delta m R^T R & \sum \delta m R^T & -\sum \delta m R^T A_{x_f} \\ \sum \delta m R & mI & -\sum \delta m A_{x_f} \\ \sum \delta m A_{x_f} R & \sum \delta m A_{x_f} & I_n \end{bmatrix} . \qquad (10\text{-}6)$$

(ii) Ponderous damping

$$[J_{ij}] = \frac{2p_p\left(\sum^{(p)} \delta m y_{pf}^{(p)\,2}\right)}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{\frac{3}{2}}} \begin{bmatrix} -(R_2 - R_1)^T A_{x_{pf}} (R_2 - R_1) & 0 & (R_2 - R_1)^T A_{x_{pf}}^2 \\ 0 & 0 & 0 \\ -A_{x_{pf}}^2 (R_2 - R_1) & 0 & -\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right) A_{x_{pf}} \end{bmatrix}$$

$$\ldots\ldots \quad (10\text{-}7)$$

(iii) Ponderous stiffnesses

$$[V_{ij}] = \frac{p_p^2\left(\sum^{(p)} \delta m y_{pf}^{(p)\,2}\right)}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^2} \begin{bmatrix} (R_2 - R_1)^T \left(\dfrac{2z_{pf}^2\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)}{\left(x_{pf}^2 + y_{pf}^2\right)^2} \begin{bmatrix} y_{pf} \\ -x_{pf} \\ 0 \end{bmatrix} \begin{bmatrix} y_{pf} & -x_{pf} & 0 \end{bmatrix} - A_{x_{pf}}^2 \right) (R_2 - R_1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\ldots\ldots \quad (10\text{-}8)$$

(iv) Gravitational

$$
[G_{ij}] = g \begin{bmatrix} 0 & 0 & -\left(\sum \delta m R^T\right) A_{\ell_{\Phi_f}} \\ 0 & 0 & -mA_{\ell_{\Phi_f}} \\ A_{\ell_{\Phi_f}}\left(\sum \delta m R\right) & 0 & -\left(\sum \delta m A_{x_f}\right) A_{\ell_{\Phi_f}} \end{bmatrix} . \qquad (10\text{-}9)
$$

(v) Propulsive

$$
\begin{aligned}
[F_{ij}] = & \begin{bmatrix} \begin{bmatrix} \frac{\partial P_f^+}{\partial x_{pf}} & \frac{\partial P_f^+}{\partial y_{pf}} & \frac{\partial P_f^+}{\partial z_{pf}} \end{bmatrix} (R_2 - R_1) & 0 & 0 \\ 0 & 0 & 0 \\ -R - \begin{bmatrix} \frac{\partial}{\partial x_{pf}} \begin{bmatrix} L_{pnf}^+ \\ M_{pnf}^+ \\ N_{pnf}^+ \end{bmatrix} & \frac{\partial}{\partial y_{pf}} \begin{bmatrix} L_{pnf}^+ \\ M_{pnf}^+ \\ N_{pnf}^+ \end{bmatrix} & \frac{\partial}{\partial z_{pf}} \begin{bmatrix} L_{pnf}^+ \\ M_{pnf}^+ \\ N_{pnf}^+ \end{bmatrix} \end{bmatrix} (R_2 - R_1) & 0 & 0 \end{bmatrix} \\
& + \frac{\left(\sum^{(p)} e_{pf}^{(p)}\right)}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{3/2}} \begin{bmatrix} R_1^T A_{x_{pf}}^2 (R_2 - R_1) & 0 & 0 \\ 0 & 0 & 0 \\ \left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right) A_{x_{pf}} R_1 + A_{x_{f1}} A_{x_{pf}}^2 (R_2 - R_1) & 0 & 0 \end{bmatrix} \\
& + \frac{\sum^{(p)} \left(x_{pf}^{(p)} e_{pf}^{(p)} + y_{pf}^{(p)} f_{pf}^{(p)}\right)}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^2} \begin{bmatrix} (R_2 - R_1)^T A_{x_{pf}}^2 (R_2 - R_1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
& + \frac{2 \sum^{(p)} y_{pf}^{(p)} f_{pf}^{(p)}}{\left(x_{pf}^2 + y_{pf}^2\right)^2 \left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)} \begin{bmatrix} (R_2 - R_1)^T \begin{bmatrix} -y^2 z^2 & xyz^2 & 0 \\ xyz^2 & -z^2 x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{pf} (R_2 - R_1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{\sum^{(p)} z_{pf}^{(p)} f_{pf}^{(p)}}{\left(x_{pf}^2 + y_{pf}^2 + z_{pf}^2\right)^{3/2}} \begin{bmatrix} -A_{x_{pf}} (R_2 - R_1) & 0 & 0 \\ 0 & 0 & 0 \\ 2A_{x_{pf}}^2 (R_2 - R_1) & 0 & 0 \end{bmatrix}
\end{aligned}
$$

...... (10-10)

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(vi) <u>Structural</u>

$$
[E_{ij}] = \begin{bmatrix} -\sum R^T \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{10-11}
$$

(vii) <u>Aerodynamic</u>

$$
-[Q_{ij}] = -\begin{bmatrix}
\sum R^T \begin{bmatrix} e_1 & \cdots & e_n \\ f_1 & \cdots\cdots & \\ g_1 & \cdots\cdots & \end{bmatrix} &
\sum R^T \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots & \end{bmatrix} D &
\sum R^T \left\{ \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} e_{\dot{\phi}} & e_{\dot{\theta}} & e_{\dot{\psi}} \\ f_{\dot{\phi}} & \cdots\cdots & \\ g_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} D \right\} \\
\begin{bmatrix} X_1 & \cdots & X_n \\ Y_1 & \cdots\cdots & \\ Z_1 & \cdots\cdots & \end{bmatrix} &
\begin{bmatrix} X_{\dot{x}} & X_{\dot{y}} & X_{\dot{z}} \\ Y_{\dot{x}} & \cdots\cdots & \\ Z_{\dot{x}} & \cdots\cdots & \end{bmatrix} D &
\begin{bmatrix} X_{\dot{x}} & X_{\dot{y}} & X_{\dot{z}} \\ Y_{\dot{x}} & \cdots\cdots & \\ Z_{\dot{x}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} X_{\dot{\phi}} & X_{\dot{\theta}} & X_{\dot{\psi}} \\ Y_{\dot{\phi}} & \cdots\cdots & \\ Z_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} D \\
\begin{bmatrix} L_1 & \cdots & L_n \\ M_1 & \cdots\cdots & \\ N_1 & \cdots\cdots & \end{bmatrix} &
\begin{bmatrix} L_{\dot{x}} & L_{\dot{y}} & L_{\dot{z}} \\ M_{\dot{x}} & \cdots\cdots & \\ N_{\dot{x}} & \cdots\cdots & \end{bmatrix} D &
\begin{bmatrix} L_{\dot{x}} & L_{\dot{y}} & L_{\dot{z}} \\ M_{\dot{x}} & \cdots\cdots & \\ N_{\dot{x}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} L_{\dot{\phi}} & L_{\dot{\theta}} & L_{\dot{\psi}} \\ M_{\dot{\phi}} & \cdots\cdots & \\ N_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} D
\end{bmatrix} .
$$

...... (10-12)

These equations differ somewhat from what might be considered to be the corresponding case among those considered in Ref 2 (see Ref 2, Table 6 "Body-fixed axes, free-free modes, displacement body freedoms"). The difference is not solely due to the different propulsive force model (section 7.4) and to the inclusion in the present development of the effect of a rotating engine which contributes the ponderous damping and ponderous stiffness terms (equations (10-6) and (10-7)). As mentioned in the Introduction, the basic cause is a slightly different representation of the deformations. Thus from equations (12) and (103) to (107) of Ref 2 it is seen that in that derivation

$$\begin{bmatrix} x_c^{(c)} \\ y_c^{(c)} \\ z_c^{(c)} \end{bmatrix} = \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} + R_0 \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + S^T \left(\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + P_q \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right) \left\{ \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + (R - R_0 + A_{x_f} P_q) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right\} . \qquad \text{...... (10-13)}$$

This agrees with the present representation, equation (2-2), as far as the first order terms, and the agreement is complete if R_0 and P_q are zero. These two matrices - matrices of modal displacements and slopes at the reference point - appeared in Ref 2 because of the indirect derivation from the equations of motion for the encastré modes case. If we put R_0 and P_q zero in Table 6 of Ref 2, the gravitational, structural and aerodynamic matrices agree* with those given above (equations (10-9), (10-11) and (10-12)). Furthermore if we put $R_2 = R_1$ and assume all the propulsive force is on a non-rotating part - this makes the two propulsive force models the same - then the only non-zero term in (10-10) is the $(- \mathit{R})$ and there is agreement with the propulsive matrix of Ref 2, Table 6, when $R_0 = P_q = 0$.

* The ponderous inertia matrices agree in any case.

Appendix

THE DERIVATIVES OF Π, Π_f

$$\Pi = \Pi(x_p, y_p, z_p) \tag{A-1}$$

$$\Pi_f = \Pi(x_{pf}, y_{pf}, z_{pf}) \tag{A-2}$$

where, *cf* equation (7-7), with

$$r^2 = x^2 + y^2 + z^2 \tag{A-3}$$

$$s^2 = x^2 + y^2 \tag{A-4}$$

$\Pi(x,y,z)$ and its required derivatives are given by

$$\Pi = \begin{bmatrix} x/r & y/r & z/r \\ -y/s & x/s & 0 \\ -xz/rs & -yz/rs & s/r \end{bmatrix} \tag{A-5}$$

$$\frac{\partial \Pi}{\partial x} = \begin{bmatrix} (r^2 - x^2)/r^3 & -xy/r^3 & -xz/r^3 \\ xy/s^3 & y^2/s^3 & 0 \\ z(x^2s^2 - y^2r^2)/(rs)^3 & xyz(r^2 + s^2)/(rs)^3 & xz^2/r^3s \end{bmatrix} \tag{A-6}$$

$$\frac{\partial \Pi}{\partial y} = \begin{bmatrix} -xy/r^3 & (r^2 - y^2)/r^3 & -yz/r^3 \\ -x^2/s^3 & -xy/s^3 & 0 \\ xyz(r^2 + s^2)/(rs)^3 & z(y^2s^2 - x^2r^2)/(rs)^3 & yz^2/r^3s \end{bmatrix} \tag{A-7}$$

$$\frac{\partial \Pi}{\partial z} = \begin{bmatrix} -xz/r^3 & -yz/r^3 & s^2/r^3 \\ 0 & 0 & 0 \\ -xs^2/r^3s & -ys^2/r^3s & -zs/r^3 \end{bmatrix} \tag{A-8}$$

$$\frac{\partial^2 \Pi}{\partial x^2} = \begin{bmatrix} 3x(x^2-r^2)/r^5 & y(3x^2-r^2)/r^5 & z(3x^2-r^2)/r^5 \\ y(s^2-3x^2)/s^5 & -3xy^2/s^5 & 0 \\ \frac{xz}{(rs)^5}\begin{Bmatrix} 3r^2s^2(r^2+s^2) \\ -x^2(3r^4+2r^2s^2+3s^4) \end{Bmatrix} & \frac{yz}{(rs)^5}\begin{Bmatrix} r^2s^2(r^2+s^2) \\ -x^2(3r^4+2r^2s^2+3s^4) \end{Bmatrix} & \frac{z^2}{r^5s^3}\{r^2s^2-x^2(r^2+3s^2)\} \end{bmatrix}$$

...... (A-9)

$$\frac{\partial^2 \Pi}{\partial y^2} = \begin{bmatrix} x(3y^2-r^2)/r^5 & 3y(y^2-r^2)/r^5 & z(3y^2-r^2)/r^5 \\ 3x^2y/s^5 & x(3y^2-s^2)/s^5 & 0 \\ \frac{xz}{(rs)^5}\begin{Bmatrix} r^2s^2(r^2+s^2) \\ -y^2(3r^4+2r^2s^2+3s^4) \end{Bmatrix} & \frac{yz}{(rs)^5}\begin{Bmatrix} 3r^2s^2(r^2+s^2) \\ -y^2(3r^4+2r^2s^2+s^4) \end{Bmatrix} & \frac{z^2}{r^5s^3}\{r^2s^2-y^2(r^2+3s^2)\} \end{bmatrix}$$

...... (A-10)

$$\frac{\partial^2 \Pi}{\partial z^2} = \begin{bmatrix} x(3z^2-r^2)/r^5 & y(3z^2-r^2)/r^5 & -3zs^2/r^5 \\ 0 & 0 & 0 \\ 3sxz/r^5 & 3syz/r^5 & s(3z^2-r^2)/r^5 \end{bmatrix} \tag{A-11}$$

$$\frac{\partial^2 \Pi}{\partial x \partial y} = \begin{bmatrix} y(3x^2-r^2)/r^5 & x(3y^2-r^2)/r^5 & 3xyz/r^5 \\ -x(3y^2-s^2)/s^5 & -y(3y^2-2s^2)/s^5 & 0 \\ \frac{yz}{(rs)^5}\begin{Bmatrix} r^4s^2 \\ -x^2(3r^4+2r^2s^2+3s^4) \end{Bmatrix} & \frac{zx}{(rs)^5}\begin{Bmatrix} r^2s^2(r^2+s^2) \\ -y^2(3r^4+2r^2s^2+3s^4) \end{Bmatrix} & -xyz^2(3s^2+r^2)/r^5s^3 \end{bmatrix}$$

...... (A-12)

$$\frac{\partial^2 \Pi}{\partial y \partial z} = \begin{bmatrix} 3xyz/r^5 & z(3y^2 - r^2)/r^5 & -y(3s^2 - 2r^2)/r^5 \\ 0 & 0 & 0 \\ xy(3s^2 - r^2)/r^5 s & -(s/r^3) + y^2(3s^2 - r^2)/r^5 s & yz(3s^2 - r^2)/r^5 s \end{bmatrix}$$

...... (A-13)

and

$$\frac{\partial^2 \Pi}{\partial z \partial x} = \begin{bmatrix} z(3x^2 - r^2)/r^5 & 3xyz/r^5 & -x(3s^2 - 2r^2)/r^5 \\ 0 & 0 & 0 \\ -(s/r^3) + x^2(3s^2 - r^2)/r^5 s & xy(3s^2 - r^2)/r^5 s & xz(3s^2 - r^2)/r^5 s \end{bmatrix} .$$

...... (A-14)

Since $\Pi\Pi^T = I$, the products $\frac{\partial \Pi}{\partial x} \Pi^T$ etc are skew symetric and we find that

$$\frac{\partial \Pi}{\partial x} \Pi^T = A_a \qquad \text{(A-15)}$$

$$\frac{\partial \Pi}{\partial y} \Pi^T = A_b \qquad \text{(A-16)}$$

$$\frac{\partial \Pi}{\partial z} \Pi^T = A_c \qquad \text{(A-17)}$$

where

$$a = \left\{ \frac{yz}{rs^2} \quad \frac{-xz}{r^2 s} \quad \frac{y}{rs} \right\} \qquad \text{(A-18)}$$

$$b = \left\{ \frac{-zx}{rs^2} \quad \frac{-yz}{r^2 s} \quad \frac{-x}{rs} \right\} \qquad \text{(A-19)}$$

$$c = \left\{ 0 \quad \frac{s}{r^2} \quad 0 \right\} . \qquad \text{(A-20)}$$

Consequently

$$\dot{\Pi}\Pi^{T} = - A_{p} \qquad \text{(A-21)}$$

with

$$p = - [a \quad b \quad c] \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

$$= - \begin{bmatrix} yz/rs^{2} & -zx/rs^{2} & 0 \\ -zx/r^{2}s & -yz/r^{2}s & s/r^{2} \\ y/rs & -x/rs & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} . \qquad \text{(A-22)}$$

The transpose of the square matrix that appears in (A-22), *ie* the matrix $\{a^{T} \quad b^{T} \quad c^{T}\}$, has been denoted by C in the main part of the report for the case when the arguments are x_{pf} etc (*cf* equation (7-15)).

A further relationship, which can be easily verified, which we will use is

$$\begin{bmatrix} a^{T} \\ b^{T} \\ c^{T} \end{bmatrix} A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Pi = \frac{A_{x}^{2}}{r^{3}} . \qquad \text{(A-23)}$$

It is easily seen also that

$$\frac{\partial \Pi}{\partial x} \frac{\partial \Pi^{T}}{\partial x} = - A_{a}^{2} \qquad \text{(A-24)}$$

$$\frac{\partial \Pi}{\partial x} \frac{\partial \Pi^{T}}{\partial y} = - A_{a} A_{b} \qquad \text{(A-25)}$$

$$\frac{\partial \Pi}{\partial x} \frac{\partial \Pi^{T}}{\partial z} = - A_{a} A_{c} \qquad \text{(A-26)}$$

$$\frac{\partial \Pi}{\partial y} \frac{\partial \Pi^{T}}{\partial y} = - A_{b}^{2} \qquad \text{(A-27)}$$

$$\frac{\partial \Pi}{\partial y} \frac{\partial \Pi^T}{\partial z} = - A_b A_c \tag{A-28}$$

$$\frac{\partial \Pi}{\partial z} \frac{\partial \Pi^T}{\partial z} = - A_c^2 \tag{A-29}$$

and so

$$\frac{\partial^2 \Pi}{\partial x^2} \Pi^T = \frac{\partial A_a}{\partial x} + A_a^2 \tag{A-30}$$

$$\frac{\partial^2 \Pi}{\partial x \partial y} \Pi^T = \frac{\partial A_a}{\partial y} + A_a A_b = \frac{\partial A_b}{\partial x} + A_b A_a \tag{A-31}$$

$$\frac{\partial^2 \Pi}{\partial x \partial z} \Pi^T = \frac{\partial A_a}{\partial z} + A_a A_c = \frac{\partial A_c}{\partial x} + A_c A_a \tag{A-32}$$

$$\frac{\partial^2 \Pi}{\partial y^2} \Pi^T = \frac{\partial A_b}{\partial y} + A_b^2 \tag{A-33}$$

$$\frac{\partial^2 \Pi}{\partial y \partial z} \Pi^T = \frac{\partial A_b}{\partial z} + A_b A_c = \frac{\partial A_c}{\partial y} + A_c A_b \tag{A-34}$$

$$\frac{\partial^2 \Pi}{\partial z^2} \Pi^T = \frac{\partial A_c}{\partial z} + A_c^2 \quad . \tag{A-35}$$

LIST OF SYMBOLS

A_{ϕ} etc	skew-symmetric matrices involving ϕ,θ,ψ etc (see equation (2-9))
A_{ij}	ponderous inertia coefficient
$B_{\phi\theta}$	see equation (2-10)
C_{u_f}	see equation (4-9)
D	differential operator d/dt
D_{ϕ}	$= \mathrm{diag}\{\phi\ \theta\ \psi\}$
$-E_i$	generalised structural force
E_{ij}	structural stiffness coefficient
$E_r^{(\mu)}, E_r^{(\mu\nu)}$	coefficient matrices in second order expression for aerodynamic forces (see equations (5-3) and (5-7))
$-G_i$	generalised gravitational force
G_i	gyrostatic force (equation (4-4))
G_{ij}	gravitational stiffness coefficient
I	unit matrix (generally 3 × 3)
I_n	$= -\ \ \delta m A_{x_f}^2$ (matrix of moments and products of inertia)
J_i	a certain coupling force between the rotational body freedoms and the deformational freedoms in Lagrange's equations referred to a non-inertial frame (see equation (4-5))
J_{ϕ}	a certain matrix formed from the elements of $\{\phi\ \theta\ \psi\}$ - see equation (3-4)
J_{ij}	ponderous damping coefficient
K_{ϕ}	$= A_{\phi} - J_{\phi}$
$K_r^{(\mu)}$	coefficient matrices in second order expression for aerodynamic forces (equation (5-7))
L	rolling moment
L_i	rolling moment coefficient (aerodynamic)
$L_{\dot{x}}, L_{\dot{\phi}}$ etc	aerodynamic rolling moment coefficients
M	pitching moment
M_i	pitching moment coefficient (aerodynamic)
$M_{\dot{x}}, M_{\dot{\phi}}$ etc	aerodynamic pitching moment coefficients
N	yawing moment

LIST OF SYMBOLS (continued)

N_i	yawing moment coefficient (aerodynamic)
P_θ	'θ factor' in axes transformation matrix (see equation (2-5))
$-P_i$	generalised propulsive force
$-P^+$	matrix of contributions to generalised propulsive force resulting from the propulsive forces on the non-rotating parts of the aircraft
P_{ij}	propulsive stiffness coefficient
Q_j	generalised force
Q_ϕ	matrix relating angular velocities and orientation (see equation (3-3))
R	modal matrix (see equation (2-1))
R_1, R_2	values of R at two points on axis of symmetry of a rotating part
R_ϕ	'ϕ factor' in axes transformation matrix (equation (2-4))
$S, S\Phi_f$	axes transformation matrix. Absence of a subscript means the arguments are ϕ, θ, ψ (see equations (2-3) and (2-8))
V_0	centrifugal potential function (see equation (4-2))
V_{ij}	ponderous stiffness coefficient
W	kinetic energy relative to the frame of reference (see equation (4-3))
X, Y, Z	overall force resolutes
X_i, Y_i, Z_i	overall force resolute coefficients (aerodynamic)
$\left.\begin{matrix} X_{\dot{x}}, X_{\dot{\phi}} \\ Y_{\dot{x}}, Y_{\dot{\phi}} \\ Z_{\dot{x}}, Z_{\dot{\phi}} \end{matrix}\right\}$ etc	aerodynamic force resolute coefficients - they may be differential operators
Y_ψ	'ψ factor' in axes transformation matrix (equation (2-6))
a_i, b_i, c_i	elements of R (equation (7-134))
e, f, g	components of local force vector
$\left.\begin{matrix} e_i, e_x, e_\phi \\ f_i, f_x, f_\phi \\ g_i, g_x, g_\phi \end{matrix}\right\}$ etc	local force vector coefficients - they may be differential operators

LIST OF SYMBOLS (continued)

$\left.\begin{array}{l} e_{\dot{x}}, e_{\dot{\phi}} \\ f_{\dot{x}}, f_{\dot{\phi}} \\ g_{\dot{x}}, g_{\dot{\phi}} \end{array}\right\}$ etc	local aerodynamic force vector coefficients - they may be differential operators
g	acceleration due to gravity
j_{Φ_f}	first column of S_{Φ_f} (see equation (6-5))
k_{Φ_f}	second column of S_{Φ_f} (see equation (6-6))
ℓ_{Φ_f}	third column of S_{Φ_f} (see equation (6-3))
m	mass of aircraft
δm	mass of a particle
n	{number of deformational degrees of freedom. {Order of symmetry of section of rotating part
p, q, r	angular velocity resolutes
p_p	angular velocity of rotating part
$p^{(p0)}, q^{(p0)}, r^{(p0)}$	see equation (7-65)
q_i	generalised coordinate
r^2	$= x^2 + y^2 + z^2$
s^2	$= x^2 + y^2$
t	time
u, v, w	linear velocity resolutes
u_m, v_m, w_m	particle velocity resolutes
x, y, z	particle position resolutes
x_1, y_1, z_1	position resolutes for the origin of the no-deformation-body-fixed axes relative to the origin of the constant-velocity axes
x_p, y_p, z_p	see equations (7-1) and (7-2)
B	matrix relating modal matrix for a general point on the rotating part with that for a point on the axis of symmetry (see equations (7-10), (7-12), (7-14) and (7-16))
$B_{\xi 1}, B_{\xi 2}, B_{\xi 3}$	see equation (7-8) *et seq*
C	see equation (7-15)
P	see equation (7-95)
P_1, P_2, P_3	see equation (7-94)
R	see equation (7-150)

LIST OF SYMBOLS (continued)

a, b, c	see equations (A-18) to (A-20)
a_{pf}, b_{pf}, c_{pf}	the columns of C^T, which are a, b, c when the arguments are x_{pf} etc
$\left.\begin{matrix} aa & ab & ac \\ ba & bb & bc \\ ca & cb & cc \end{matrix}\right\}$	the elements of BP^T (see equations (7-115) and (7-116))
Θ_f	angle of inclination in datum motion
Π	product of 'θ and φ factors' in axes transformation matrix for transformation from the no-deformation-body-fixed axes to the rotating part-'fixed' axes (see equation (7-7))
Φ_f	angle of bank in datum motion
Ψ_f	nose-azimuth angle in datum motion
thus	
Φ_f, Θ_f, Ψ_f	are the orientation angles of the constant-velocity axes relative to the normal earth-fixed axes
Π	see equation (A-5)
α, β, γ	see equations (7-50) and (7-50b)
ξ, η, ζ	elements of arbitrary vector
ϕ, θ, ψ	orientation angles of no-deformation-body-fixed axes relative to constant-velocity axes
diag $\{\xi, \eta, \zeta\}$	the diagonal matrix $\begin{bmatrix} \xi & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \zeta \end{bmatrix}$
$\sum$	indicates summation for all the particles of the aircraft
$\sum^{(p)}$	indicates summation over the rotating part
$\sum^{(p+)}$	indicates summation over the total region where the propulsive force acts
$\sum^{(p+)-(p)}$	$= \sum^{(p+)} - \sum^{(p)}$

LIST OF SYMBOLS (concluded)

DRESSINGS

(i) Subscripts

A quantity is relative to or about the origin of a particular set of axes where

- c denotes the constant-velocity axes
- n denotes the no-deformation-body-fixed axes
- p denotes axes 'fixed' in the rotating part

f indicates the value during the datum motion

g indicates gravitational

p indicates propulsive

s indicates structural

Occasionally a subscript is added to an array to show that all the elements of the array should have that subscript.

(ii) Superscripts

Bracketed superscripts denote the axes of resolution with the same significance as in (i) above. Also

T denotes the transpose of a matrix

\+ indicates a contribution from the non-rotating parts of the aircraft where propulsive forces act

(p0) see equation (7-65)

(iii) Suprascripts

• (dot) denotes derivative with respect to time

‒ (bar) denotes typical or total

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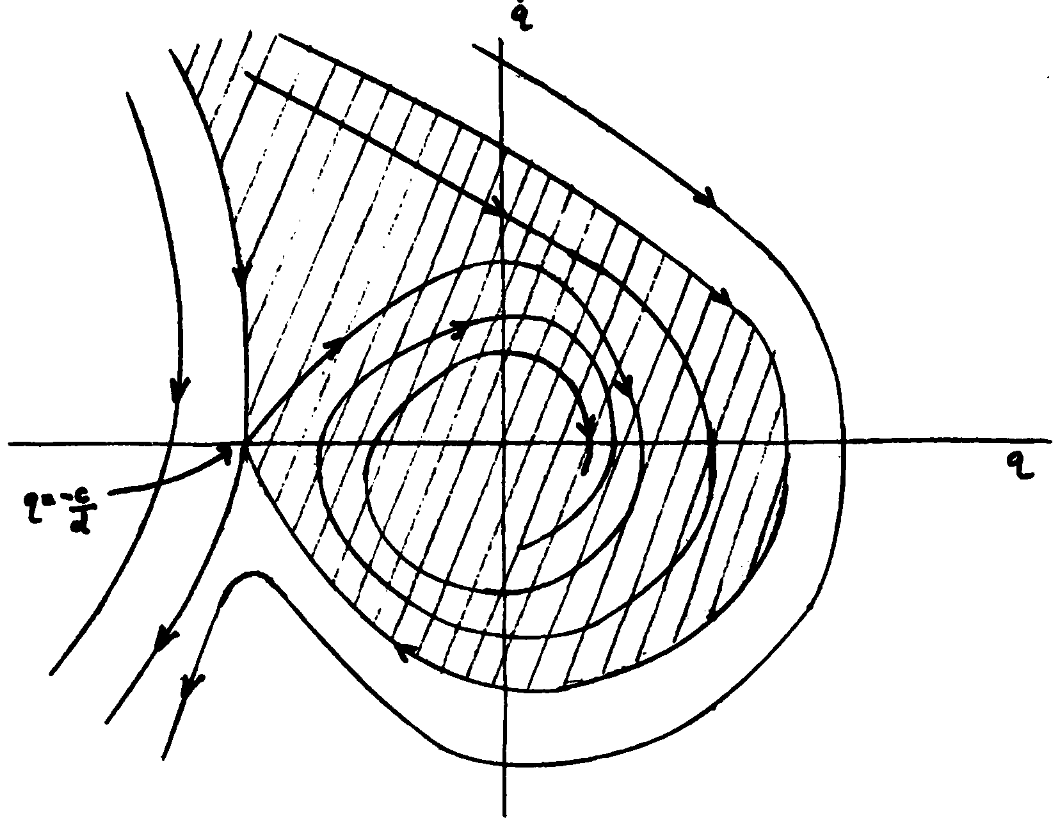


Fig 1. $\ddot{q} + b\dot{q} + cq + dq^2 = 0 \qquad b, c > 0$

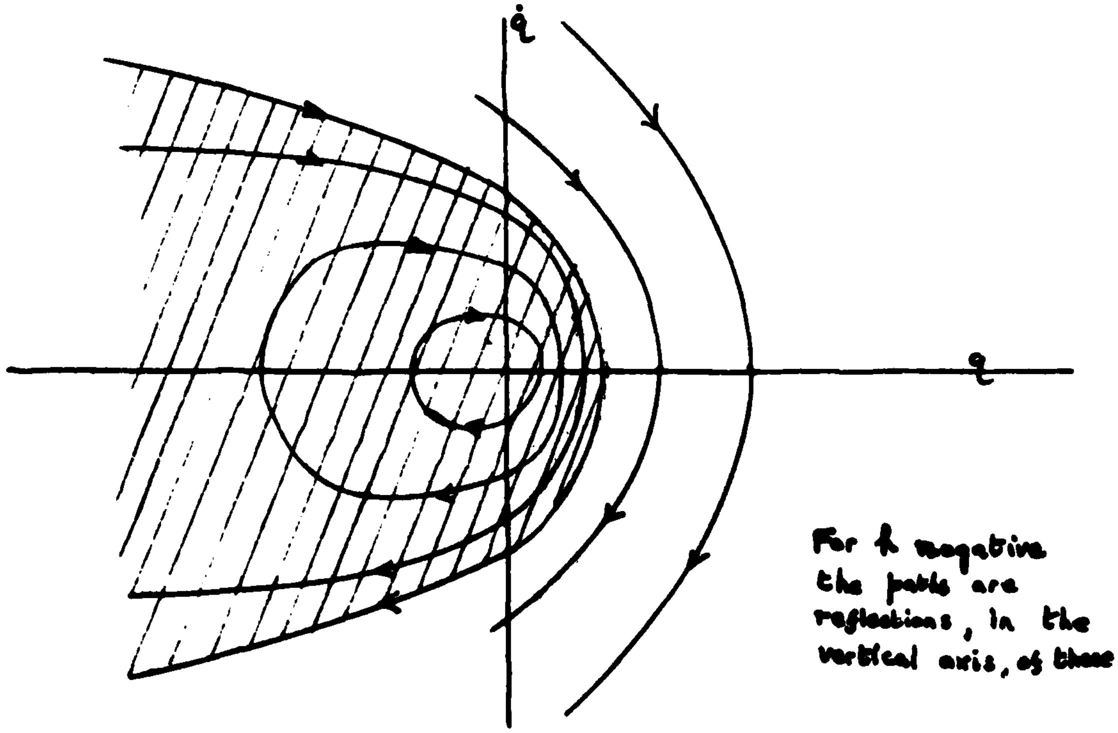
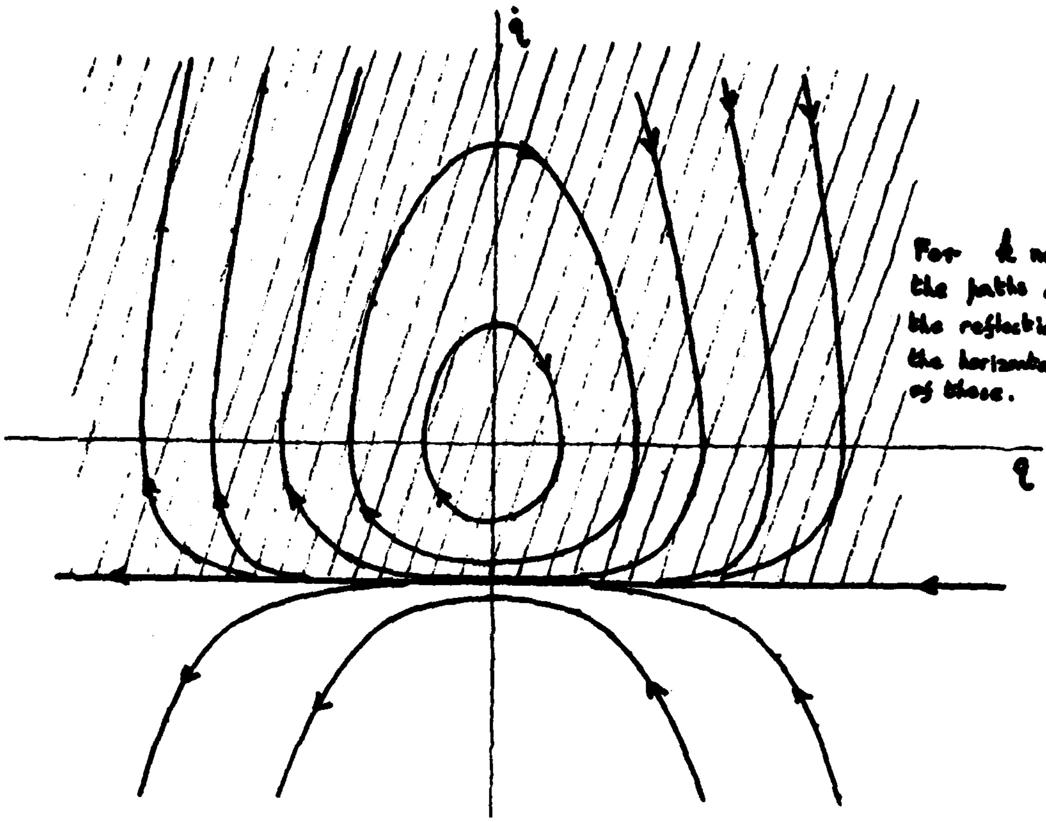


Fig 2. $\ddot{q} + b\dot{q}^2 + cq = 0 \qquad b, c > 0$

Figs 3&4



For k negative the paths are the reflections, in the horizontal axis, of these.

Fig 3. $\ddot{q} + kq\dot{q} + cq = 0 \qquad k, c > 0$

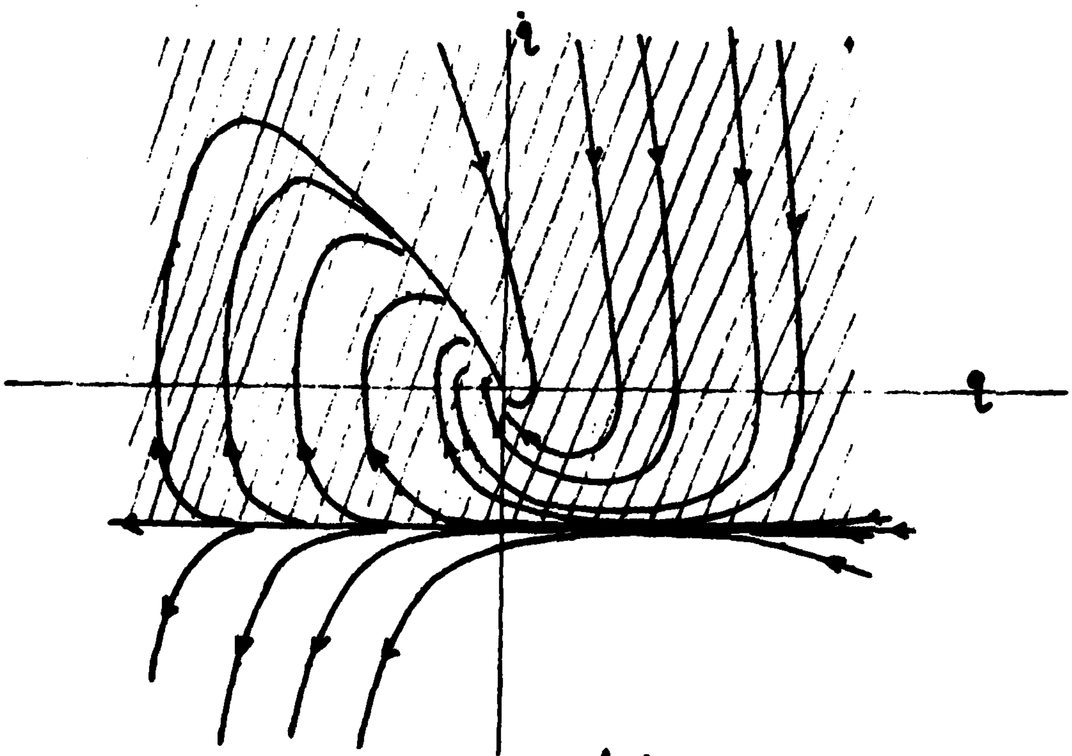


Fig 4. $\ddot{q} + (b\dot{q} + cq)(1 + \frac{k}{c}q) = 0 \qquad k, b, c > 0$

REPORT DOCUMENTATION PAGE

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1. DRIC Reference (to be added by DRIC)	2. Originator's Reference	3. Agency Reference	4. Report Security Classification/Marking
	RAE TM Structures 941	N/A	UNLIMITED

5. DRIC Code for Originator	6. Originator (Corporate Author) Name and Location
7673000W	Royal Aircraft Establishment, Farnborough, Hants, UK
5a. Sponsoring Agency's Code	**6a. Sponsoring Agency (Contract Authority) Name and Location**
N/A	N/A

7. Title Formulation of the equations of motion of a deformable aircraft using Lagrange's equations in an arbitrary non-inertial frame of reference

7a. (For Translations) Title in Foreign Language

7b. (For Conference Papers) Title, Place and Date of Conference

8. Author 1. Surname, Initials	9a. Author 2	9b. Authors 3, 4	10. Date	Pages	Refs.
Woodcock, D.L.	-	-	December 1978	99	14
11. Contract Number	**12. Period**	**13. Project**	**14. Other Reference Nos.**		
N/A	N/A				

15. Distribution statement
(a) Controlled by –
(b) Special limitations (if any) –

16. Descriptors (Keywords) (Descriptors marked * are selected from TEST)

Aeroelasticity. Dynamics. Non-linear. Equations of motion.

17. Abstract

The equations of motion of a deformable aircraft are developed in detail from Lagrange's equations for a non-inertial frame. Particular account is taken of the influence of the propulsive and effective forces produced by power units containing rotating parts. The development ventures to a certain extent into the non-linear regime. By an appropriate choice of deformation modes the principal frame of reference can be specified as, for example, mean-body axes or body-fixed axes.

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